Rendiconti per gli Studi Economici Quantitativi

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Efficient Monte Carlo pricing of portfolio options

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Abstract. Monte Carlo (MC) methods can be used to price derivatives for which closed evaluation formulas are not available or difficult to derive. A drawback of the method can be its high computational cost, especially if applied to basket options, whose payoffs depend on more than one asset.

This article presents a kind of control variate to reduce the variance of estimated prices, by replacing terminal values of stocks with their unconditional expectations in the payoff function. We apply the variance reduction method to some portfolio options, achieving in some case remarkable speed and accuracy in price estimation.

KEY WORDS: option pricing, Monte Carlo methods, variance reduction, basket options.

AMS Classification: 65C05, 91B28; *JEL Classification*: C15,G12.

1 Introduction

The valuation of complex options produced a vast literature in last years. Among the methods used when close evaluation formulas are missing or difficult to derive there are tree and partial differential equation (PDE) methods. The former approximate the unknown distribution of payoffs discretizing the jumps in the value of the underlying asset, similarly to the binomial model, Cox et al. (1979). The latter solve the numerical partial differential equation satisfied by the price of the option. In many important cases, there is not a close solution (like in the Black–Scholes (BS) paper, Black and Scholes (1973)) and numerical techniques are employed (see Wilmott (1995) for a PDE introduction to option pricing).

* I would like to thank A. Basso, P. Pianca that introduced me to the subject and an anonymous referee. J. Topper pointed me some errors and provided helpful discussion. Some preliminary work on basket and exotic options was funded by Cray Research S.P.A. and Department of Mathematics and Computer Science, University of Venice.

This paper deals with Monte Carlo (MC) pricing methods. In Cox and Ross (1976) it is noted that if a riskless hedge can be formed, the option value is the risk-neutral discounted expectation of its payoffs. Hence the price can be estimated by MC methods, simulating many independent paths of the underlying assets and taking the discounted mean of the generated payoffs. In principle, this can be done even if complex distributions or payoffs are involved, provided that we know the path generating process of the assets (that are commonly thought to be the realization of a lognormal random walk).

Since the seminal paper by Boyle (1977) on the application of MC methods to option pricing, it was realized that refinement of the methods was desirable, being the accuracy (i.e. standard deviation) of the estimates of the order of $1/\sqrt{N}$. This unfortunately means that to double the precision we have to multiply by four the number of simulations N and the time needed for computation. Hence some variance reduction techniques are introduced. Among the papers dealing with variance reduction of the estimates of options prices there are Boyle (1977), Kemna and Vorst (1990), Clewlow and Carverhill (1993).

This paper aims to propose a variance reduction technique for MC pricing of portfolio options, whose payoff is a function of more than one asset. In particular, we will define some control variates that can reduce the number of simulations needed to achieve satisfactory precision. To our knowledge, little work has been done on this subject although variance reduction methods appears particularly useful when multivariate random variables are generated in simulation, with increased computational cost.

It is implied in some papers that propose tree-based methods that MC methods are not suitable for multivariate option pricing. We feel that this is somewhat misleading and that the use of proper variance reduction schemes can greatly enhance the performance of MC methods, producing in some cases great speed and accuracy.

In section 2 we present the basic idea of control variates to reduce variance of estimates. Section 3 presents the basic control variates, that are essentially obtained transforming the payoffs to a function of one single asset. This usually allows to evaluate the mean of the control variate with a modification of BS formula. In particular we replace some asset with their unconditional mean in the payoff function. We show the effects in variance reduction of the previously defined control variates in section 4, where prices of options are calculated. We provide also some comparisons with quasi-Monte Carlo pricing, an emerging area in Monte Carlo methods. Finally, some concluding remarks are given in section 5.

2 The variance reduction idea

In this section we briefly describe the use of control variates to reduce variance of Monte Carlo estimates. A detailed treatment of the subject is in Ripley (1987) and Hammersley and Handscomb (1967). We do not discuss the antithetic variates, that could be implemented with little effort.

Suppose we are interested in estimating the expectation of the random variable X, and we are given the independent sample $\{X_1, \ldots, X_N\}$ extracted form the distribution of X. The natural unbiased estimator is the sample mean

$$\hat{M} = \frac{1}{N} \sum_{i=1}^{N} X_i. \tag{1}$$

Suppose, moreover, that we can generate from the distribution of the control variate Y the independent control variates $\{Y_1, \ldots, Y_N\}$ simultaneously with the X_i's. Then

$$\hat{M}_Y = \frac{1}{N} \sum_{i=1}^{N} (X_i - Y_i + E[Y]) \tag{2}$$

is still an unbiased estimator of the expectation of X. If we compare the variance of $\hat{M}$ and $\hat{M}_Y$ we get

$$\mathrm{Var}(\hat{M}_Y) = \mathrm{Var}(\hat{M}) + \frac{1}{N} \left(\mathrm{Var}(Y) - 2\,\mathrm{Cov}(X,Y) \right). \tag{3}$$

We have that $\mathrm{Var}(\hat{M}_Y) \leq \mathrm{Var}(\hat{M})$ provided that

$$\mathrm{Cov}(X,Y) \geq \frac{\mathrm{Var}(Y)}{2}. \tag{4}$$

Hence, if

$$\mathrm{Corr}(X,Y) \geq \frac{1}{2} \sqrt{\frac{\mathrm{Var}(Y)}{\mathrm{Var}(X)}},$$

the estimator $\hat{M}_Y$ has smaller variance than $\hat{M}$ and is preferable in Monte Carlo simulation.

From a practical point of view the definition of a suitable control variate Y needs some care to give strong positive correlation with X and easy evaluation of the mean $E[Y]$ that appears in (2). These are often contrasting targets, and it might be difficult to find simultaneously strong correlation with X and close analytical formula for $E[Y]$.

Example 1. Consider a portfolio option on two assets that pays at time T the (random) sum

$$S = f(S_{1T}, S_{2T}) = \max\{0, q_1 S_{1T} + q_2 S_{2T} - k\},$$

where S_{1T}, S_{2T} are the values at time T of assets S_1, S_2, k is the strike price and q_1, q_2 are the quantities of assets in the portfolio. A control variate that can be considered is

$$Y = f(E[S_{1T}], S_{2T}) = \max\{0, q_1 E[S_{1T}] + q_2 S_{2T} - k\}.$$

Y is obviously correlated with S and the expectation $E[Y]$ can be evaluated in close form, being Y the payoff of a call option on the asset S_2.

In the next section we introduce a set of control variates, that appears to be widely applicable to basket option pricing.

3 Mean variance reduction

The following model and notation are standard in financial papers and details can be found, for example, in Wilmott et al. (1995). Consider assets $S_1, \ldots, S_n$, with normally distributed logarithmic returns, standard deviations $\sigma_1, \ldots, \sigma_n$ and that pay dividends at continuously compounded rate $d_1, \ldots, d_n$ respectively. Let $\mu_i = r - d_i - \sigma_i^2/2, i = 1, \ldots, n$, where r is the risk-free (istantaneous) rate of the market, and assume we want to price at time $t = 0$ an european-like asset that pays the sum

$$C_T = f(S_{1T}, \ldots, S_{nT}), \tag{5}$$

at time T, where S_{iT} denotes the value of i-th asset at maturity T. The initial value of the i-th asset will be denoted by s_{i0}.

Under fairly standard assumptions on the the risk neutrality of agents, the price of (5) can be estimated by generating many realizations of $\{S_{1T}^{(j)}, \ldots, S_{nT}^{(j)}\}, j = 1, \ldots, N$ and discounting the sample mean of the resulting $\{C_T^{(j)} = f(S_{1T}^{(j)}, \ldots, S_{nT}^{(j)}), j = 1, \ldots, N\}$, providing

$$\hat{C}_T = e^{-rT} \frac{1}{N} \sum_j C_T^{(j)}. \tag{6}$$

This might well be a hard computational task, as many vector random variables are to be generated from a multivariate distribution. Even more importantly, the standard deviation of the estimated price is $O(1/\sqrt{N})$ and hence a huge N might be required to achieve satisfactory precision.

Let us describe a simple implementation of variance reduction scheme based on control variates. Recall from section 2 that a candidate control variate is a random variable possibly correlated with $C_T^{(j)}$ and such that its mean value is available. Consider the following *unconditional mean* control variates $UM_T(i), i = 1, \ldots, n$:

$$UM_T(i) = f(E\left[S_{1T}\right], \ldots, E\left[S_{i-1,T}\right], S_{iT}, E\left[S_{i+1,T}\right], \ldots, E\left[S_{nT}\right]). \tag{7}$$

The variate $UM_T(i)$ is obtained from (5) replacing S_{jT} with its unconditional mean $E\left[S_{jT}\right]$ if $i \neq j$. It is obvious that $UM_T(i)$ is generally correlated with C_T and $E[UM_T(i)]$ can be easily evaluated in many important cases (i.e. if f assumes a specific functional form).

Example 2. Assume we have to price a portfolio options on three assets, with payoff

$$f(S_{1T}, S_{2T}, S_{3T}) = \max\{0, q_1 S_{1T} + q_2 S_{2T} + q_3 S_{3T} - k\}.$$

Then the three control variates $UM_T(i), i = 1, 2, 3$ are

$$UM_T(1) = \max\{0, q_1 S_{1T} + q_2 E\left[S_{2T}\right] + q_3 E\left[S_{3T}\right] - k\}, \tag{8}$$
$$UM_T(2) = \max\{0, q_1 E\left[S_{1T}\right] + q_2 S_{2T} + q_3 E\left[S_{3T}\right] - k\}, \tag{9}$$
$$UM_T(3) = \max\{0, q_1 E\left[S_{1T}\right] + q_2 E\left[S_{2T}\right] + q_3 S_{3T} - k\}. \tag{10}$$

The previous assumptions on the distribution of logarithmic returns of assets S_{1T}, S_{2T}, S_{3T} yields

$$\begin{aligned} E\left[S_{1T}\right] &= s_{10} \exp\left(\mu_1 T + \frac{\sigma_1^2 T}{2}\right), \\ E\left[S_{2T}\right] &= s_{20} \exp\left(\mu_2 T + \frac{\sigma_2^2 T}{2}\right), \\ E\left[S_{3T}\right] &= s_{30} \exp\left(\mu_3 T + \frac{\sigma_3^2 T}{2}\right). \end{aligned} \tag{11}$$

The means of the control variates $UM_T(i), i = 1, 2, 3$ are readily evaluated. Note, for example, that the right hand side of (10) can be rewritten as

$$q_3 \max\{0, S_{3T} - \frac{k - q_1 E\left[S_{1T}\right] - q_2 E\left[S_{2T}\right]}{q_3}\}$$

which we recognize easily as the payoff of an european call option on the asset S_3 and strike price $K_3 = (k - q_1 E\left[S_{1T}\right] - q_2 E\left[S_{2T}\right])/q_3$. Hence the expected value can be derived by the Black–Scholes formula

$$\begin{aligned} &E\left[UM_T(3)\right] = \\ &q_3 e^{rT} \left[s_{30} \exp(-d_3 T) N(p_1) - \frac{k - q_1 E\left[S_{1T}\right] - q_2 E\left[S_{2T}\right]}{q_3} e^{-rT} N(p_2)\right], \end{aligned} \tag{12}$$

where

$$\begin{aligned} p_1 &= \frac{\log\left[q_3 s_{30} / (k - q_1 E\left[S_{1T}\right] - q_2 E\left[S_{2T}\right])\right] + (r - d_3 + \sigma_3^2/2) t}{\sigma_3 \sqrt{T}}, \\ p_2 &= p_1 - \sigma_3 \sqrt{T}, \end{aligned}$$

and $N(\cdot)$ is the cumulative normal distribution.

For the general portfolio option on n assets we have

$$E\left[UM_T(i)\right] = q_i e^{rT} \left[s_{i0} \exp(-d_i T) N(p_1) - \frac{K_i}{q_i} e^{-rT} N(p_2)\right], \quad i = 1, \ldots, n, \tag{13}$$

with strike price

$$K_i = k - \sum_{j=1, j \neq i}^{n} q_j E\left[S_j\right]$$

and where

$$p_1 = \frac{\log\left[q_i s_{i0}/K_i\right] + (r - d_i + \sigma_i^2/2)t}{\sigma_i\sqrt{T}}, \quad p_2 = p_1 - \sigma_i\sqrt{T}.$$

The control variates (7) allow us to obtain a set of Monte Carlo estimates $\hat{C}_T^i$ of the unknown price $\hat{C}_T$:

$$\hat{C}_T^i = e^{-rT}\frac{1}{N}\sum_{j=1}^{N} C_T^{(j)} - UM_T^{(j)}(i) + E\left[UM_T(i)\right], \quad i = 1, \ldots, n \quad (14)$$

where $UM_t^{(j)}(i)$ denotes the $j-$th realization of the control variate $UM_t(i)$.

Indeed, as there is no reason to use just one of the n control variates at a time, we can obtain other estimates of the unknown price: for example we can define

$$\hat{C}_T^{i_1,i_2} = e^{-rT}\frac{1}{N}\sum_{j=1}^{N} C_T^{(j)} - UM_T^{(j)}(i_1) - UM_T^{(j)}(i_2) + E\left[UM_T(i_1)\right] + E\left[UM_T(i_2)\right],$$

and a moment of reflection shows that $2^n - 1$ reduced variance prices are available. As far as we know, this is one of the few cases in which an increase in the dimension n of the problem naturally brings some good news, doubling the number of control variates for each dimension.

In the following we will consider mainly relations (14) and

$$\hat{C}_T^{all} = e^{-rT}\frac{1}{N}\sum_{j=1}^{N}\left[C_T^{(j)} - \sum_{i=1}^{n} UM_T^{(j)}(i) + \sum_{i=1}^{n} E\left[UM_T(i)\right]\right],$$

which is in a sense the most we can do to reduce variability, having used at the same time *all* the relations (14).

The following result, from Ross (1997), gives some information on the effect of one of the previously defined control variates.

Proposition 1. *Let $X_1, \ldots, X_n$ be a vector of independent random variables and let the functions $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ be not decreasing in each component. Then*

$$E\left[f(X_1, \ldots, X_n)g(X_1, \ldots, X_n)\right] - E\left[f(X_1, \ldots, X_n)\right]E\left[g(X_1, \ldots, X_n)\right] \geq 0,$$

that is $f(X_1, \ldots, X_n)$ and $g(X_1, \ldots, X_n)$ are positively correlated.

If f is the payoff function and we set

$$g(S_{1T},\dots,S_{nT}) = UM_T(i) = \\ f(E[S_{1T}],\dots,E[S_{i-1,T}],S_{iT},E[S_{i+1,T}],\dots,E[S_{nT}]),$$

then the above theorem allows to conclude that the control variate $UM_T(i)$ is effective in reducing the variance of the estimated price, provided that the asset prices $S_1,\dots,S_n$ are independent random variables.

Unfortunately, the previous result is not applicable when assets are correlated, which is quite common in financial markets. The following proposition gives sufficient conditions for positive correlation of payoff $f(S_{1T},\dots,S_{nT})$ and $UM_T(i)$ in the presence of correlated assets.

Proposition 2. *Let Σ be the positive definite covariance matrix of the logarithmic returns of the assets $S_1,\dots,S_n$ and assume that the payoff function f is not decreasing in each component. Then, if the Cholesky decomposition L of the matrix Σ is (componentwise) nonnegative, we have:*

$$E[f(S_{1T},\dots,S_{nT})UM_T(i)] - \\ E[f(S_{1T},\dots,S_{nT})]\,E[UM_T(i)] \geq 0,$$

that is, $f(S_{1T},\dots,S_{nT})$ and $UM_T(i)$ are positively correlated.

Proof. Observe that if a matrix L is nonnegative then $\mathbf{z} \mapsto L\mathbf{z}$ is a non decreasing function in each variable. It is trivial to verify that if $\mathbf{z}$ is a vector of independent standard normal random variables, $L\mathbf{z}$ is a multinormally distributed vector with covariance matrix Σ. The random payoff $f(S_{1T},\dots,S_{nT})$ can be written as,

$$f\left(\exp\left(s_{10}L_1\cdot(z_1,\dots,z_n)\right),\dots,\exp\left(s_{n0}L_n\cdot(z_1,\dots,z_n)\right)\right),$$

where $\cdot$ denotes the scalar product, the z_i's are independent normal random variables and L_j is the j-th row of the matrix L. It is now immediate to check that f satisfy the hypothesis of theorem 1 as a function of the independent z_i's, being a composition of the non decreasing functions $\mathbf{z} \mapsto L\mathbf{z}$, $z \mapsto \exp z$, $\mathbf{z} \mapsto f(\mathbf{z})$. ∎

Note that we can find examples of nonnegative Σ that give raise to Cholesky decompositions with *negative* elements, when dimension

$n \geq 3$. Hence, even with pairwise positive correlated assets we cannot generally guarantee that a particular control variate alone is reducing variance.

Before showing some numerical examples, note that the key idea of this section was that the replacement of S_{iT} with its *unconditional* mean produces the payoff function of a standard european option which is easily priced. It would be interesting to investigate on the use the use of *conditional* expectations which could potentially be rewarding when returns are significantly correlated.

4 Application to portfolio options

In this section we apply the variance reduction method described in the previous section to some portfolio options, for which there is no close pricing formula. In these cases the use of a MC method provides an estimate of the value of the option, together with the sample standard deviation to assess the precision of the result.

It is customary to test the MC pricing procedure against analytical pricing formulas. This turns out to be quite difficult with portfolio options as there is no known close formula for $n \geq 2$. However, as a benchmark, we can consider an exchange option Margrabe (1978) on two assets whose payoff[1] is

$$f(S_{1T}, S_{2T}) = \max\{0, S_{2T} - S_{1T}\}, \tag{15}$$

and for which the following analytic pricing formula is available

$$C_T = s_{20}e^{-d_2 T}N(p) - s_{10}e^{-d_1 T}N(p - \Sigma\sqrt{T}), \tag{16}$$

where $N(\cdot)$ is the cumulative standard normal distribution and

$$p = \frac{\log\left(\frac{s_{20}e^{-d_2 T}}{s_{10}e^{-d_1 T}}\right)}{\Sigma\sqrt{T}} + \frac{1}{2}\Sigma\sqrt{T}, \quad \Sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2. \tag{17}$$

[1] An exchange option is a special case of portfolio option, with $k = 0, q_1 = -1, q_2 = 1$. The underlying "portfolio" is a combination of a long position on S_2 and a short position on S_1. In particular, being some $q_i < 0$, the use of the *put* Black-Scholes formula is necessary for variance reduction. This is not the case for the standard portfolio with $q_i \geq 0$, $\forall i$. Finally, note that an exchange option is also a special case of a spread option with general payoff $f(S_1, S_2) = \max(0, S_2 - S_1 - k)$.

Table 1. Estimated prices and relative standard deviations for plain and reduced variance Monte Carlo.

	Plain MC		Red. C^1		Red. C^2		Red. C^{all}	
N	$\hat{C}$	std	$\hat{C}$	std	$\hat{C}$	std	$\hat{C}$	std
1000	16.76	0.72	16.27	0.43	16.36	0.47	15.87	0.16
2000	15.77	0.50	15.95	0.30	15.81	0.32	15.99	0.11
3000	16.05	0.41	16.14	0.25	15.91	0.26	16.00	0.09
4000	16.03	0.36	16.15	0.21	15.94	0.23	16.06	0.08
5000	16.01	0.32	16.02	0.19	16.02	0.20	16.03	0.07
6000	16.05	0.29	16.01	0.17	16.02	0.19	15.99	0.06
7000	15.98	0.27	16.01	0.16	15.99	0.17	16.01	0.06
8000	15.93	0.25	16.01	0.15	15.96	0.16	16.04	0.05
9000	15.93	0.23	16.00	0.14	15.98	0.15	16.05	0.05
10000	16.06	0.22	16.09	0.13	16.03	0.14	16.06	0.05
true	16.0606							

Setting $s_{10} = s_{20} = 100, r = \log(1.1), d_1 = d_2 = \log(1.05), \sigma_1 = 0.3, \sigma_2 = 0.2, \rho = -0.5$ and $T = 0.95$ we get the price 16.0606 from (16). Table 1 shows some estimates obtained by plain MC and variance reduction with relative standard deviation for different sample sizes. It is apparent that variance reduction techniques produce an error 4 to 5 times smaller than plain Monte Carlo methods. This means that, given a predetermined precision, the evaluation of the price can be obtained 16 to 25 times faster and the variance reduced estimate with $N = 1000$ is preferable to the result with $N = 10000$ naive simulations. Figure 1 depicts the estimated prices against N and the true price. A look at the plot shows that the reduced variance estimates are smoothly converging to the true price, while the plain Monte Carlo fluctuate widely around the proper price. Note also that C^{all} has a much smaller standard deviation than C^1 or C^2: this happens in all the simulations we performed on exchange and portfolio options.

Next, we examine a portfolio option on 3 assets. We assume the returns have the following correlation matrix

$$S = \begin{bmatrix} 1 & \rho & \frac{1}{2}\rho \\ \rho & 1 & \frac{1}{4}\rho \\ \frac{1}{2}\rho & \frac{1}{4}\rho & 1 \end{bmatrix}.$$

Fig. 1. Estimates of exchange option price with plain and variance reduction Monte Carlo.

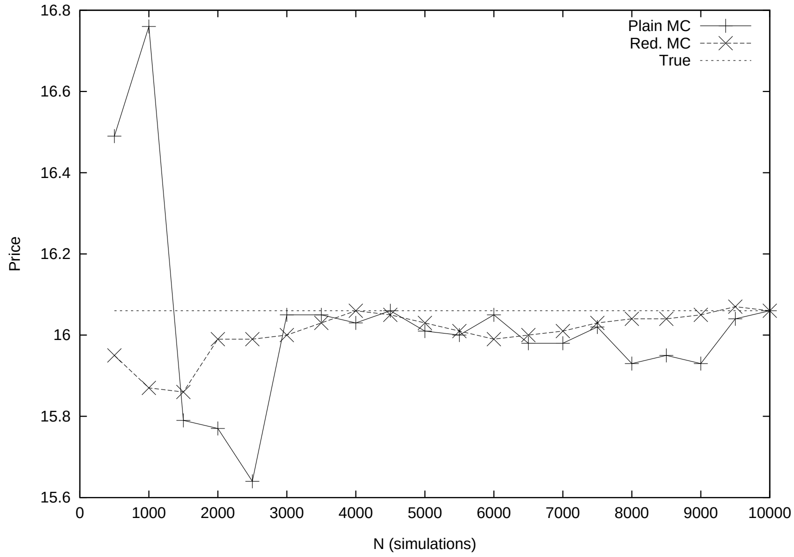


Table 2 shows the estimated prices with relative standard deviations for plain and variance reduced (C^{all}) for different values of the correlation parameter ρ and of σ_1. The ratio of variances is a measure of the reduction in computer time needed to obtain an estimate, given a predetermined accuracy (i.e. standard deviation). A look at the table shows that prices are computed 3 to 15 times faster if C^{all} is used.

Table 2. Results of Monte Carlo pricing of a portfolio option ($n = 3$). For each value of σ_1 and ρ estimated prices C^{Plain} and C^{all} (with standard deviation in brackets) are listed. We set $s_{01} = s_{02} = s_{03} = 100, k = 300, T = 0.95, r = \log(1.05), d_1 = \log(1.01), d_2 = \log(1.02), d_3 = \log(1.03), \sigma_2 = 0.2, \sigma_3 = 0.3, N = 100000$.

Portfolio ($n = 3$)			
$\sigma_1 \backslash \rho$	0.5	0.0	-0.5
0.1	20.59 (0.094)	18.26 (0.082)	15.48 (0.069)
	20.66 (0.029)	18.34 (0.029)	15.57 (0.028)
0.2	23.30 (0.108)	19.83 (0.089)	15.36 (0.067)
	23.30 (0.033)	19.87 (0.035)	15.40 (0.036)
0.3	26.31 (0.125)	22.09 (0.101)	16.61 (0.072)
	26.23 (0.036)	22.07 (0.040)	16.59 (0.043)

We then study a portfolio option on 10 assets. For simplicity, we assume their returns are uncorrelated ($S = I_{10}$) and no dividend is continuously distributed ($d_i = 0, i = 1, \ldots, 10$). We fixed all the parameters but strike price k and q_1, the quantity of the first stock in the portfolio. This allows us to evaluate the performance of the pricing algorithm also for deeply in and out of the money options. The following Table 3 exhibits the results we get, for plain and reduced variance Monte Carlo (C^{all}).

Table 3. Results of Monte Carlo pricing of a portfolio option ($n = 10$). For each value of q_1 and strike k estimated prices C^{Plain} and C^{all} (with standard deviation in brackets) are listed. We set $T = 0.95, r = \log(1.05), s_{0i} = 10, d_i = 0, \sigma_i = 0.1 + 0.02i,\ i = 1, \ldots, 10, q_2 = \ldots = q_{10} = 1, N = 10000$.

Portfolio ($n = 10$)			
$q_1 \backslash k$	100	110	120
1	5.52 (0.055)	0.97 (0.025)	0.07 (0.007)
	5.46 (0.022)	0.97 (0.022)	0.07 (0.007)
2	14.60 (0.070)	5.94 (0.058)	1.15 (0.028)
	14.56 (0.004)	5.90 (0.022)	1.15 (0.024)
3	24.56 (0.075)	15.06 (0.074)	6.42 (0.062)
	24.53 (0.003)	15.03 (0.004)	6.39 (0.023)

As far as interpretation is concerned, the table can be split in three parts. The upper diagonal entries, corresponding to out of the money options, show no speed improvement in price estimation. This is a well known problem, that basically affects many pricing techniques for deeply out of the money options. The diagonal prices are evaluated 5 to 8 times faster if reduced MC is used. The lower diagonal terms show the best performance, being prices evaluated hundreds (300 to 600) of times faster than with plain MC.

We conclude this section with three pricing examples in which quasi-Monte Carlo simulation (with and without unconditional mean variance reduction) is used. In particular, we employ Sobol quasi-random numbers and we refer the interested reader to Boyle et al. (1997) and Joy et al. (1996) for further financial applications of quasi-Monte Carlo methods.

In order to compare different Monte Carlo methods we provide the root mean squared pricing percentage error (in the following, for brevity, percentage error), which is evaluated as

$$100\sqrt{\frac{1}{R}\sum_{i=1}^{R}\left(\frac{C-\hat{C}_i}{C}\right)^2},$$

where $\hat{C}_i$ is the estimated price in the i-th run, R denotes the number of independent pricing runs for each option and C is the true price. We set $R = 100$, i.e. the percentage error is averaged on 100 estimated prices.

Fig. 2. Percentage error of computed price for exchange option of Table 1, whose theoretical price is 16.06, for different Monte Carlo methods.

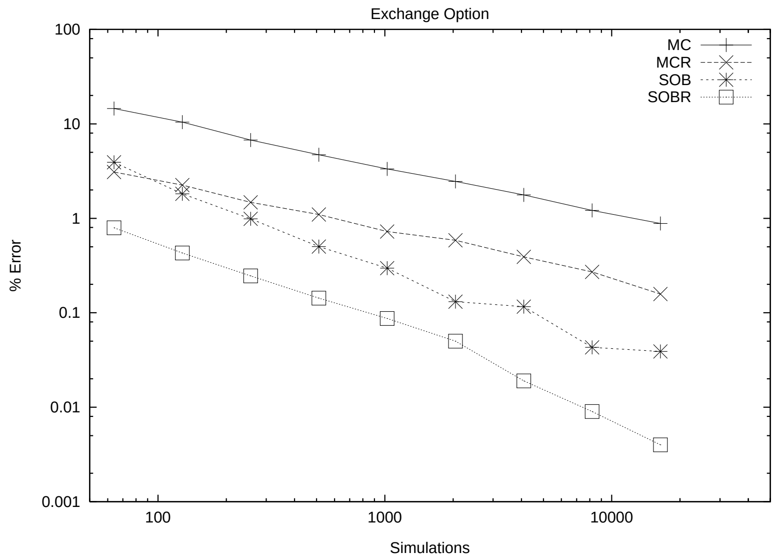


Figure 2 show the percentage error of the exchange option of Table 1 against the number of simulations. The graph shows the performance of plain Monte Carlo (MC), variance reduced Monte Carlo (MCR), Sobol quasi-Monte Carlo (SOB) and variance reduced Sobol quasi-Monte Carlo (SOBR). It is apparent that the variance reduction method is equally effective with pseudo and quasi-Monte Carlo methods. The use of Sobol sequences together with variance reduction allows to get about 1% error in less than 100 simulations. The same error is achieved by using about 1000 "plain" Sobol simulations.

Fig. 3. Percentage error of computed price for low volatility 5-dimensional portfolio call option. The reference price is 3.163.

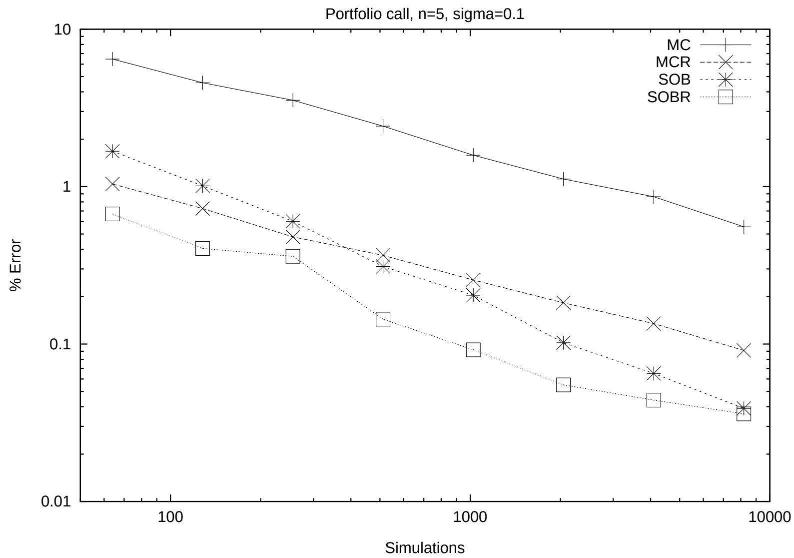
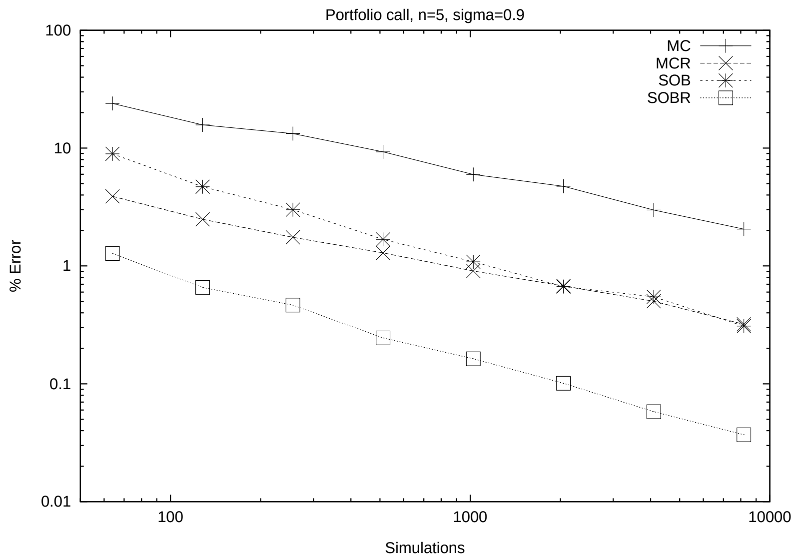


Fig. 4. Percentage error of computed price for high volatility 5-dimensional portfolio call option. The reference price is 16.328.



Figures 3 and 4 are intended to compare the efficacy of the method on two 5-dimensional portfolio call options, respectively written on uncorrelated, low volatility assets and on highly correlated, high volatility assets. The parameters we used are the following: $s_{i0} = 10, q_i = 1, \sigma_i = 0.1, d_i = \log(1 + i/100), i = 1, \ldots, 5, r = \log(1.1)$, time to expiration $T = 1$, no correlation among assets and strike $k = 50$ for the first option; $s_{i0} = 10, q_i = 1, \sigma_i = 0.9, d_i = \log(1 + i/100), i = 1, \ldots, 5, r = \log(1.1)$, time to expiration $T = 1$, pairwise correlation among assets $\rho = 0.75$ and strike $k = 50$ for the second option. As there is no close formula to evaluate the price C of such portfolio options, we use as benchmark price for the evaluation of the percentage error the price obtained by 100.000 variance reduced Sobol simulations. The computed prices are respectively 3.163 and 16.328.

Both pictures show that unconditional mean variance reduction is effective in reducing percentage error in both pseudo and quasi-Monte Carlo simulations. In the low volatility case, we obtain 5 times smaller error with variance reduced pseudo Monte Carlo, given the number of simulations. This reduction factor drops to about 2 for the quasi-Monte Carlo simulation. In the high volatility case of Figure 4 the error can be reduced about 9 times for both methods, given the number of simulations. Moreover, for this particular case, plain Monte Carlo with variance reduction is more accurate than quasi-Monte Carlo if the number of simulation is smaller than 10.000.

Note that these comparisons are solely meant to show that unconditional mean variance reduction is effective for different sampling schemes, like pseudo and quasi-random numbers. Some experiences by the author show that the method is equally efficient also for moment matching methods and quadratic resampling, but this will be reported in future papers.

5 Conclusion

Some conclusive remarks are the following. In general, the use of control variates appears to reduce considerably the standard deviation of estimated prices. Given a fixed accuracy, computations appear to run on average from 5 to 10 times faster than plain pseudo MC methods. The results on some high dimensional portfolio options are encouraging, and reduced variance MC produces accurate prices with remarkable speed.

We compare our results with quasi-Monte Carlo estimates: the proposed method appear to be effective in this case too, providing estimates that are 2 to 9 times more accurate.

Other simulations are obviously necessary to draw more general conclusions but the method we described appears to be effective on the many examples we tried, of easy extension to other basket derivatives, like dual or spread options and applicable to a variety of sampling schemes.

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