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On Efficient Trading Mechanisms with Ex-Post Individually Rational Traders

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Abstract. This paper studies a bilateral trading setting where the two agents are not ex-ante identified, in the sense that each of them may end up being a net buyer or a net seller. We derive a sufficient condition that ensures the existence of an (ex-post) efficient, (ex-post) budget balanced, (interim) incentive compatible trading mechanism that always yields a positive net utility to all agents (ex-post individually rational). This result improves a former existence result based on interim individual rationality showing that the stronger requirement of ex-post individual rationality does not always rule out efficiency.

Keywords: mechanism design, bilateral trade, ex-post individual rationality, Groves mechanism.

JEL Classification: D02, D40, D44, D82.

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1 Introduction

A central question that the mechanism design literature tries to address is the design of institutions that optimally allocate goods among privately informed traders. In fact, whenever the number of agents is small and the assumption of complete information does not hold, the strategic behavior of the agents might prevent the economy at hand from achieving the competitive equilibrium outcome, which is well known to be Pareto efficient. Therefore, one of the objectives of mechanism design is to find institutions that allow to achieve Pareto efficiency even when the assumptions on which the competitive equilibrium rests are not satisfied; or institutions that are in some sense optimal even when we are in presence of some form of market failure.

In addition, the study of mechanisms provides insights on the process of price formation, on which the classical theory of competitive equilibrium is somewhat silent. The theory of competitive equilibrium says that prices adjust automatically to clear markets, as if they were driven by some “invisible hand”. Thus, studying trading mechanisms might help explain what are the actual forces that drive prices towards the equilibrium, and whether different market structures have different implications in the process of price adjustment.

Since the seminal paper of Myerson and Satterthwaite [8], the literature on optimal trading mechanisms has grown rapidly and a number of results have been achieved. In their work, Myerson and Satterthwaite show that, in the presence of incomplete information about traders’ valuations, an Ex-Post Efficient (henceforth Ex-Post EFF), Ex-Post Budget Balanced (Ex-Post BB), Bayesian Incentive Compatible (Bayesian IC)¹, Interim Individually Rational (Interim IR) mechanism to allocate an indivisible good between a buyer and a seller does not in general exist. Given this impossibility result, they characterize second best mechanisms, i.e. mechanisms that are BB, IC, IR and maximize the joint *expected* welfare (Ex-Ante EFF). Their analysis is carried on assuming linear utilities. The only condition is that the supports of the distributions of types have a nonempty intersection, i.e. it is not common knowledge that there are positive gains from trade. The impossibility result by Myerson and Satterthwaite has been then generalized to the case of quasilinear utilities and multiple buyers/sellers².

Cramton, Gibbons and Klemperer [1] have demonstrated that the impossibility result by Myerson and Satterthwaite [8] rests crucially on the hypothesis that the agents are ex-ante *identified*, i.e. it is commonly known which is the (potential) seller and which is the (potential) buyer. In Myerson and Satterthwaite this is done by simply assuming that the only good to be

¹Bayesian IC is equivalent to Interim IC. In the following, Bayesian and Interim IC will be used interchangeably.

²See Williams [10].

traded is initially owned by one of the agents, so that trade can take only one direction. However, when the agents are not ex-ante identified, i.e. when it is not commonly known who is going to be the seller and who is going to be the buyer, the impossibility result is replaced by a possibility result: under proper conditions, it is possible to design an Ex-Post BB, Bayesian IC, Interim IR trading mechanism that *always* gives rise to an efficient allocation. The hypothesis of *unidentified* traders is introduced by simply assuming that the only good to be traded is initially jointly owned by the agents. Cramton, Gibbons and Klemperer show that, in an environment with n agents, linear utilities and equally distributed types, efficiency can be achieved provided that traders have equal or nearly equal initial endowments. Notice that in their paper the problem is formulated in terms of partnership: every agent initially owns a share of a single asset and the objective is to dissolve this partnership efficiently. However, the result holds also when we consider the problem of designing an efficient trading mechanism between a number of agents, each endowed with some quantity of a homogeneous good.

The subsequent literature has tried to generalize and enrich these two fundamental results in several directions. The main purpose of this strand of literature was to characterize other possibility/impossibility results. Clearly, the conclusions one gets are strictly related to the number and the types of constraints one introduces. Moreover, it has been argued that a desirable mechanism should not be too sensitive to the particular environment under investigation, i.e. it should be “robust”; very often, a trade-off arises between robustness and efficiency.

Makowski and Mezzetti [6] unify the impossibility result of Myerson and Satterthwaite [8] and the possibility result of Cramton, Gibbons and Klemperer [1] by providing necessary and sufficient conditions for the existence of (1) an Ex-Post EFF, Ex-Post BB, Bayesian IC, Interim IR mechanism; and of (2) an Ex-Post EFF, Ex-Ante BB, Dominant Strategy IC³, Ex-Post IR mechanism. Their analysis considers a quite general environment (n traders, quasilinear utilities, general distributions of types).

Williams [10] generalizes the Myerson and Satterthwaite [8] bilateral trading problem to a multilateral setting with m buyers and n sellers. He shows that the existence of an Ex-Post EFF, Ex-Ante BB, Bayesian IC, Interim IR mechanism depends upon the relative sizes of m and n and upon the supports and distributions of valuations. These two last papers make extensive use of the equivalence between efficient, incentive compatible mechanisms and Groves mechanisms.

Hagerty and Rogerson [3] focus on robust trading mechanisms, where by robust they mean mechanisms that are Dominant Strategy IC and Ex-Post IR. Such mechanisms are robust in the sense that they are independent of agents’ beliefs. For the bilateral trading setting of Myerson and Satterthwaite

³Dominant Strategy IC is equivalent to Ex-Post IC.

[8], they show that the only mechanisms that satisfy Ex-Post BB, Dominant Strategy IC and Ex-Post IR are posted price mechanisms, which are clearly inefficient.

Recently, Schweizer [9] has summarized the sufficient conditions for the existence of an Ex-Post EFF, Ex-Post BB, Bayesian IC, Interim IR trading mechanism, for any prior distribution of valuations. Other significant contributions in this field include Gresik [2], Mookherjee and Reichelstein [7], Kosmopoulou and Williams [5], Kosmopoulou [4].

2 Objective

We could briefly summarize the literature on this topic saying that, for the case of identified traders, a possibility or impossibility result has been established for any conceivable set of requirements.

On the other hand, when traders are unidentified, i.e. every trader might end up being a net buyer or a net seller, there is still at least a gap to be filled: to our knowledge, nothing has yet been said on the possibility/impossibility of designing a mechanism that is simultaneously Ex-Post EFF, Ex-Post BB, Interim IC, Ex-Post IR. In fact, Cramton, Gibbons and Klemperer [1] have established a possibility result when IR is required to hold only at the Interim stage. And Makowski and Mezzetti [6] have shown an impossibility result when IC is required to hold Ex-Post. We put ourselves in between and ask ourselves: *When agents are not ex ante identified, is it possible to design an Efficient trading mechanism that is Ex-Post Budget Balanced, Interim Incentive Compatible and Ex-Post Individually Rational?* Answering to this question is the aim of this paper.

The last requirement is what distinguishes our approach from most of the existing literature. In Cramton, Gibbons and Klemperer [1], the Individual Rationality constraint is required to hold only at the Interim stage. This means that each trader can decide whether or not to participate in the mechanism, but this decision has to be made without knowing others' private information. In that framework, an agent will decide to take part in the mechanism whenever his *expected* net payoff from participating is non-negative. However, it may well happen that the *actual* net payoff he gets, i.e. his *ex-post* net payoff, is negative. On the other hand, *Ex-Post* IR allows a trader to learn the terms of trade before deciding whether or not to accept the deal and this rules out the possibility that the agent ends up with a negative net payoff; hence, no trader will ever regret the choice he made. This makes Ex-Post IR much more appealing than Interim IR. In fact, in most markets each trader has the ability to refuse to trade when the negotiated terms give him negative net utility. And even if the designer is a government agency with some power to bind traders to the outcomes emerging from the mechanism, the designer may want to avoid costly court battles to enforce

them. Moreover, if agents have limited liability, an Ex-Post IR mechanism is the only viable alternative. However, Ex-Post IR is a more restrictive constraint than Interim IR and thus Ex-Post IR mechanisms constitute a subset of Interim IR mechanisms. Is this subset empty?

The interest in this issue is thus both technical and substantial: on the one hand, it enlarges the range of possibility/impossibility results for trading mechanisms. On the other hand, an efficient trading mechanism that is also Ex-Post IR is more “robust” than an Interim IR one, and is thus preferable.

3 The model

The model we adopt is standard in the literature. Consider the following social choice problem: there are 2 agents, each agent i is endowed with a quantity s_i of some homogeneous good. Let us normalize the s_i 's so that $s_1 + s_2 = 1$. In other words, in the economy there is a total endowment of one unit of a good and each agent owns a share of it. The initial endowments are common knowledge.

Let A be the set of all the possible allocations of the good among the agents and let $a = (a_1, a_2)$ be a general element of A , where a_i represents the share allocated to agent i . Clearly, $a_1 + a_2 = 1$. Each agent i has quasi-linear preferences over the set of allocations and money given by the utility function:

$$u_i(a, t_i; v_i) = \pi_i(a; v_i) + t_i$$

where t_i is any money transfer to agent i . The parameter $v_i \in V_i$ is the type of agent i and is private information. It can be interpreted as the valuation of the good to the agent. However, it is commonly known that types are independent random variables and that v_i has marginal distribution F_i with strictly positive density f_i over its support.

The social choice problem consists in finding a trading mechanism that, in equilibrium, allocates the good among the agents in some desirable manner.

As it is well known from the revelation principle, any Bayesian equilibrium outcome of any conceivable mechanism can be obtained as the equilibrium outcome of a direct mechanism in which players truthfully report their types. Thus, there is no loss of generality in restricting our attention to direct mechanisms.

A direct mechanism is a pair of functions $\{a, t\}$, where $a : V_1 \times V_2 \rightarrow A$ is an allocation function and $t = (t_1, t_2) : V_1 \times V_2 \rightarrow \mathbb{R}^n$ is a transfer function.

We look for a direct mechanism where truthful reporting is an equilibrium, i.e. we look for an Incentive Compatible direct mechanism.

Moreover, we want that the mechanism be consistent with voluntary participation, in other words we want it to be Individually Rational. This requirement appears appropriate in this setting, as it is consistent with the

idea that an agent cannot be forced to trade; in other social choice problems, e.g. provision of a public good, this is not necessarily the case.

Typically, there are two additional properties that a desirable trading mechanism should satisfy: one is Efficiency (the allocation maximizes joint welfare), the other is Budget Balance (there's no need to subsidize agents). This last property is natural in this context of exchange of goods among private agents.

In this framework, time plays a crucial role. The sequence of events can be divided in three temporal stages: at the *ex-ante* stage, each agent knows only the distribution of types of all the agents; at the *interim* stage, each agent has learned his own type but is still uncertain about the other agents' types; at the *ex-post* stage, the types of all the agents are commonly known. The behavioral assumptions and the properties that the mechanism should satisfy can be defined accordingly.

The three temporal stages are increasingly restrictive: a mechanism that satisfies an *ex-post* requirement will satisfy, *a fortiori*, also the corresponding *interim* requirement; and a mechanism that satisfies an *interim* requirement will satisfy, *a fortiori*, the corresponding *ex-ante* requirement. The converse is in general not true. The obvious implication is that a possibility result still holds when the requirements are relaxed to a previous temporal stage; correspondingly, an impossibility result immediately extends to the following temporal stages.

Incentive Compatibility

A direct mechanism $\{a, t\}$ determines an allocation and a vector of transfers as a function of the reports of the agents. As we said before, by invoking the revelation principle, we can restrict our attention to IC direct mechanisms where truthtelling is an equilibrium. Now, according to the notion of equilibrium adopted, one gets different IC constraints. In particular, we focus on Interim (or Bayesian) IC, that is we require that honest reporting be a Bayesian equilibrium of the game induced by the mechanism. In symbols, for any agent i :

$$E_{-i}[u_i(a(v_i, v_{-i}), t_i(v_i, v_{-i}); v_i)] \geq E_{-i}[u_i(a(r_i, v_{-i}), t_i(r_i, v_{-i}); v_i)], \quad \forall v_i, \forall r_i$$

where E_{-i} means expectation taken with respect to the distributions of other agents' types and r_i is the report by agent i .

Individual Rationality

An IC direct trading mechanism $\{a, t\}$ is Ex-Post IR if:

$$u_i(a(v), t_i(v); v_i) \geq u_i^0(v), \quad \forall i, \forall v,$$

where $u_i^0(v)$ is the outside option to agent i (i.e. the utility he gets by not participating in the mechanism and keeping his initial share s_i).

If instead the above inequality holds only in expectation, we have the milder requirement of Interim IR:

$$E_{-i}[u_i(a(v), t_i(v); v_i)] \geq E_{-i}[u_i^0(v)], \quad \forall i, \forall v_i.$$

Efficiency

An incentive compatible direct trading mechanism $\{a, t\}$ is Ex-Post EFF if the allocation function maximizes the total welfare of the economy, whatever the profile of types happens to be. In symbols:

$$a(v) \in \arg \max_{a \in A} \sum_{k=1}^2 \pi_k(a, v_k).$$

In the following, we assume that the above program has a solution. To avoid introducing further notation, $a(v)$ will denote such a solution.

Budget Balance

A direct trading mechanism $\{a, t\}$ is Ex-Post BB if the transfers balance in any possible state of the world:

$$\sum_{k=1}^2 t_k(v) = 0, \quad \forall v.$$

Once we find an allocation function $a(v)$ that is Ex-Post EFF, our problem reduces to finding a transfer function $t(v)$ that simultaneously satisfies Ex-Post BB, Interim IC and Ex-Post IR. Through the equivalence between IC, EFF mechanisms and Groves mechanisms (see Makowski and Mezzetti [6]), we can reformulate this problem in an equivalent, though more tractable manner.

A Groves mechanism in expectations is a mechanism $\{a, t\}$ such that: (1) a is Ex-Post EFF and (2) the trading charges $h_1(v), h_2(v)$ are lump sums in expectations, that is $\forall i, \forall v_i, v'_i$:

$$E_{-i}[h_i(v_i, v_{-i})] = E_{-i}[h_i(v'_i, v_{-i})] = H_i, \quad (1)$$

where the trading charges are defined as follows:

$$h_i(v) = \sum_{k=1}^2 \pi_k(a(v), v_k) - \pi_i(a(v), v_i) - t_i(v).$$

For our purposes, it might be useful introducing the following notation: let

$$g(v) = \sum_{k=1}^2 \pi_k(a(v), v_k)$$

be the total welfare generated by the allocation $a(v)$. Being $a(v)$ the Ex-Post EFF allocation, $g(v)$ represents the maximum welfare that can be achieved in the economy. The expectation of $g(v)$ taken with respect to all agents and to the other agent will be denoted by G and $G_i(v_i)$ respectively; that is,

$$G = E[g(v)], \quad G_i(v_i) = E_{-i}[g(v)].$$

With this new notation, the trading charges are:

$$h_i(v) = g(v) - \pi_i(a(v), v_i) - t_i(v), \quad i = 1, 2,$$

and ex-post utilities are:

$$u_i(v) = \pi_i(a(v), v_i) + t_i(v) = g(v) - h_i(v), \quad i = 1, 2. \quad (2)$$

Equation (2) says that a Groves mechanism in expectations is a direct mechanism that allocates the good efficiently and that gives each agent a utility level equal to the total (maximal) welfare of the economy minus a non distortional trading charge.

The equivalence result states that, under independence of types, a mechanism is Ex-Post EFF and Bayesian IC if and only if it is a Groves mechanism in expectations⁴.

The idea behind this equivalence is the following. By giving to each agent the total welfare generated by trade (which is the maximum achievable when the allocation is efficient), the private benefit of each agent coincides with the social benefit, and this guarantees that the agent will act accordingly to the interest of the society. By subtracting a non-distortional charge, that is a charge whose expected value is independent on an agent's true type, the agent's incentive to tell the truth is not affected.

Let us now introduce the additional requirements of BB and IR. In terms of Groves mechanisms, Ex-Post BB becomes:

$$h_1(v) + h_2(v) = g(v), \quad \forall v. \quad (3)$$

Interim IR becomes:

$$H_i \leq G_{-i}(v_i) - E_{-i}[u_i^0(v)], \quad \forall i, \forall v_i, \quad (4)$$

while Ex-Post IR becomes:

$$h_i(v) \leq g(v) - u_i^0(v), \quad \forall i, \forall v. \quad (5)$$

⁴See Makowski and Mezzetti [6], Theorem 2.2, page 505.

The basic Groves mechanism is a Groves mechanism where the non-distortionary charges are set equal to zero, i.e. $h_1(v) = h_2(v) = 0, \forall v$. In such a mechanism each agent is simply given the total welfare generated by trade, i.e. $u_1(v) = u_2(v) = g(v)$. Clearly, this mechanism generates a budget deficit, given by $g(v)$ (each agent's ex-post utility is $g(v)$, but the total welfare that can be distributed is only $g(v)$). To cover this budget deficit, it is sufficient to impose positive charges to the agents. These charges must of course be non-distortionary, in order not to violate Bayesian IC. However, by increasing those charges, we run the risk of violating IR. It is precisely this trade-off between BB and IR that strictly constrains the possibility of finding a Groves mechanism that also satisfies BB and IR. The IR constraints impose an upper bound to the trading charges that could make impossible to raise enough money to cover the deficit.

4 Interim Individual Rationality

In the environment considered by Myerson and Satterthwaite [8], where BB is required to hold Ex-Post while IR only at the Interim stage, such a trade-off precludes the possibility of finding a Groves mechanism that simultaneously satisfies BB and IR. In that framework, IR implies that the seller cannot be taxed at all, and the maximum charge that can be imposed to the buyer is not enough to finance the deficit.

Cramton, Gibbons and Klemperer [1], instead, have shown that, when agents are not identified, and when initial endowments are sufficiently evenly distributed, the upper bounds on trading charges imposed by Interim IR are not strong enough to prevent from achieving BB.

Makowski and Mezzetti [6] have obtained a necessary and sufficient condition for the existence of an Ex-Post EFF, Ex-Post BB, Interim IC and Interim IR mechanism for a general mechanism design problem. This condition encompasses both the result of Myerson and Satterthwaite [8] and that of Cramton, Gibbons and Klemperer [1]. The condition for the two agents' case is that:

$$G \leq \sum_{k=1}^2 \inf_{v_k} (G_{-k}(v_k) - E_{-k}[u_k^0(v)]) \quad (6)$$

(see also Williams [10] and Schweizer [9]).

Condition (6) can be interpreted as follows: on the left hand side we have the expected value of the budget deficit generated by a basic Groves mechanism, that is a Groves mechanism with no extra charges. To cover such a deficit, we introduce trading charges $h_1(v), h_2(v)$. These trading charges have to be non distortionary, i.e. their expectations evaluated with respect to other agents' distributions of types, must be constants (equation (1)). Moreover, the trading charge imposed to the i -th agent must not violate the corresponding Interim IR, that is its expected value cannot exceed

the expected value of the net utility agent i gets from participating in the mechanism (equation (4)). But since H_i is a constant, we obtain the following upper bound on agent i 's expected trading charge:

$$H_i \leq \inf_{v_i} (G_{-i}(v_i) - E_{-i}[u_i^0(v)]).$$

The sum of these upper bounds represents the maximum amount (in expected terms) that can be extracted from the agents without violating their Interim IR constraints. If such an amount is sufficient to cover the expected budget deficit of a basic Groves mechanism, then a mechanism with the desired properties does exist.

5 Ex-Post Individual Rationality: a Sufficient Condition for existence

Since Ex-Post IR is more restrictive than Interim IR, condition (6) is still necessary but not sufficient for the existence of an Ex-Post EFF, Ex-Post BB, Interim IC and Ex-Post IR mechanism.

Ex-Post IR (condition (5)) imposes a tighter upper bound on the trading charges that can be levied to ensure budget balance. One might expect that such an upper bound is so strong a constraint that precludes existence. However, this is not always the case. In Proposition 1, which is the main result of this paper, we obtain a sufficient condition for the existence of the desired mechanism.

PROPOSITION 1: *Let $a(v)$ be an Ex-Post Efficient allocation and let $g(v)$ denote the total welfare generated by such an allocation when v is the realized profile of types. If, for all agents $i = 1, 2$ and for every profile of types v ,*

$$\sum_{k=1}^2 \inf_v \left\{ \frac{1}{2}g(v) + \frac{1}{2} \left(G_k(v_k) - G_{3-k}(v_{3-k}) \right) - u_k^0(v) \right\} \geq 0 \quad (7)$$

then there exists an Ex-Post BB, Interim IC and Ex-Post IR trading mechanism that implements $a(v)$.

Proof. The proof is by construction. Let $a(v)$ be an Ex-Post EFF allocation function. Our objective is to find trading charges, $h_1(v)$ and $h_2(v)$, that simultaneously satisfy (i) Ex-Post BB (condition (3)), (ii) Interim IC (condition (1)), (iii) Ex-Post IR (condition (5)).

We proceed as follows: we first construct a two-parameter set of functions $h_1(v)$ and $h_2(v)$ that always satisfy Interim IC. Then, within this set, we optimally determine the value of the parameters in order to meet the Ex-Post IR constraint. If these parameters are compatible with Ex-Post BB, then existence is guaranteed.

Now, consider the following families of functions:

$$h_i(v) = \frac{1}{2}g(v) - \frac{1}{2}\left(G_i(v_i) - G_{3-i}(v_{3-i})\right) + C_i, \quad i = 1, 2. \quad (8)$$

where C_i is a constant. Equation (8) defines two families of functions parameterized by the constants C_1 and C_2 . These functions satisfy Interim IC for all C_1 and C_2 . In fact, for all i and for all v_i :

$$E_{-i}[h_i(v)] = \frac{1}{2}G_i(v_i) - \frac{1}{2}\left(G_i(v_i) - G\right) + C_i = \frac{1}{2}G + C_i,$$

a constant.

The trading charges $h_1(v)$ and $h_2(v)$ satisfy Ex-Post IR (condition(5)) if and only if:

$$C_i \leq \frac{1}{2}g(v) + \frac{1}{2}\left(G_i(v_i) - G_{-i}(v_{-i})\right) - u_i^0(v), \quad \forall i, \forall v. \quad (9)$$

Now, for $i = 1, 2$, define:

$$C_i^* = \inf_v \left\{ \frac{1}{2}g(v) + \frac{1}{2}\left(G_i(v_i) - G_{-i}(v_{-i})\right) - u_i^0(v) \right\}.$$

Ex-Post IR is thus satisfied if and only if $C_i \leq C_i^*$.

Finally, Ex-Post BB (condition (3)) requires that $C_1 + C_2 = 0$. Therefore, Ex-Post IR and Ex-Post BB can be simultaneously satisfied if and only if $C_1^* + C_2^* \geq 0$, which completes the proof. ■

In the proof of Proposition 1, the trading charges $h_1(v)$ and $h_2(v)$ (equation (8)) are constructed as follows. The first term, $g(v)/2$, is the budget deficit generated by the basic Groves mechanism equally divided between the two agents. This term is independent of i . The quantity between round brackets is the difference between the two agents' expected total welfare. In a sense, the budget deficit is first equally spread among the agents; but then, an agent with higher than the average expected gains from trade is acquainted a discount while an agent with lower than the average expected gains from trade receives an additional charge. This additional term gives the agent the right incentives to tell the truth, without affecting budget balance. Finally, a lump-sum term C_i is added or subtracted. Broadly speaking, the aim of this lump-sum is to extract the maximum possible informational rent from the trader which is in a better position. This sum is then transferred to the other trader.

To see this, consider the trading charges corresponding to the case when the lump-sum terms are equal to zero, that is:

$$\tilde{h}_i(v) = \frac{1}{2}g(v) - \frac{1}{2}\left(G_i(v_i) - G_{3-i}(v_{3-i})\right), \quad i = 1, 2.$$

With these trading charges, agent i 's net utility is

$$\frac{1}{2}g(v) + \frac{1}{2}\left(G_i(v_i) - G_{-i}(v_{-i})\right) - u_i^0(v).$$

Now, suppose that, for some agent, this quantity happens to be strictly positive for all realizations of types, whereas the other agent's net utility is negative for some realization of types. Then it is possible for the designer to apply an additional charge to the first agent to extract his informational rent. In order not to violate Interim IC, this additional charge must be constant. In order not to violate Ex-Post IR, this charge cannot exceed the net utility of the agent in the worst case scenario (equation (9)). This lump-sum can then be transferred to the other agent. If this transfer is sufficient to meet his Ex-Post IR constraint, we have an existence result.

6 Uniform types

Suppose that valuations are independent and uniformly distributed, with partially overlapping supports. That is, $v_1 \sim U[0, 1]$ and $v_2 \sim U[a, a + 1]$, with $a \in [0, 1)$. Moreover, suppose traders are risk neutral with linear utilities: $u_i(v) = s_i a_i(v) + t_i(v)$, $i = 1, 2$.

In this case, the ex-post efficient allocation is the one that assigns the whole good to the agent with the highest valuation, i.e.:

$$a_1(v_1, v_2) = \begin{cases} 1 & \text{if } v_1 \geq v_2 \\ 0 & \text{otherwise} \end{cases}, \quad a_2(v_1, v_2) = 1 - a_1(v_1, v_2)$$

The sufficient condition in Proposition 1 is satisfied as long initial endowments are:

$$s_1 = \frac{1-a}{2}, \quad s_2 = \frac{1+a}{2}. \quad (10)$$

In particular, when the supports of valuations are equal ($a = 0$), we get the condition that initial endowments have to be perfectly symmetric ($s_1 = s_2 = 1/2$); in this case, $C_1^* = C_2^* = 0$ and the trading charges are:

$$h_i(v) = \frac{1}{2} \max\{v_i, v_j\} - \frac{1}{4}(v_i^2 - v_j^2) \quad i = 1, 2.$$

Notice that, with Interim IR, the (necessary and sufficient) condition that guarantees the existence of the Ex-Post EFF mechanism is that the initial endowments must be *sufficiently* symmetric; more precisely: $s_1 \in [1/2 - \sqrt{3}/6, 1/2 + \sqrt{3}/6]^5$. Here, instead, existence is guaranteed if endowments are *exactly* symmetric⁶. In other words, making the IR constraint more restrictive cause the interval of endowments that ensure existence to shrink

⁵See Makowski and Mezzetti [6].

⁶It is worth reminding that our condition is only sufficient.

to a single point, that is the center of that interval. The reason is the following: when individual rationality must hold ex-post, the lump-sum part of the trading charges cannot exceed the *worst case scenario* net *ex-post* utility of the agent in the absence of lump sum charges (equation (9)). Now, an agent with a low endowment has a net utility equal to zero in the worst case scenario, which corresponds to the situation in which both agents have null valuations for the good. In this case, the mechanism would assign a worthless good to one of the agent, without requiring any payment. Instead, an agent with high endowment has a strictly negative worst case scenario net payoff: the worst case scenario to him is the situation in which both agents have the highest possible valuation, which would cause his trading charge to exceed the utility he gets from acquiring the good. Clearly, the only situation in which the sum of the two agents' worst case scenario net payoffs is nonnegative is the one associated to perfectly symmetric endowments.

The strategic reasoning behind the impossibility result of Myerson and Satterthwaite [8] is basically this: a buyer has an incentive to undervalue his willingness to pay, while a seller has an incentive to overvalue. These incentives go in opposite directions and are so strong that can make trade impossible even in situations where trade would be profitable. When agents are not identified, instead, they don't know whether they will be ex-post buyers or sellers. Hence, the two opposite forces compensate each other (at least partially), and this lowers the incentives necessary to induce the agents to tell the truth.

These strategic forces are clearly at work in the result of equation (10), which says that the initial endowments of the agent with "low" valuation (agent 1) must be lower than that of the agent with "high" valuation (agent 2). The reason is the following: agent 1 knows that his own valuation is likely to be lower than agent 2's valuation. Thus, he expects that, with high probability, he will be the seller because the other agent is likely to be more eager to get the good than he is. On the other hand, he has a low initial endowment of the good, and thus, if he gets the whole good, he will get utility on a large fraction of the good. This makes him more willing to buy than to sell. Therefore, there are two forces that go in opposite directions: the first pushes agents 1 to behave more like a seller, while the second leads him to behave more like a buyer. The opposite happens for agent 2. When initial endowments are given by (10), these two forces compensate and agents are willing to tell the truth.

7 Concluding remarks

In this paper we consider a bilateral trading framework with privately informed agents where traders are not ex ante identified, i.e. it is not commonly known who is the buyer and who is the seller. We start from the

existence result first obtained by Cramton, Gibbons and Klemperer [1] and then generalized by Makowski and Mezzetti [6], which says that an Ex-Post Efficient, Ex-Post Budget Balanced, Bayesian Incentive Compatible and Interim Individually Rational trading mechanism might exist, provided that the environment satisfies a particular condition. In the case of linear utilities and uniformly distributed types, the condition is that agents have equal or nearly equal initial endowments.

We show that, contrary to what one would expect, if we strengthen the Individually Rational constraint from Interim to Ex-Post, at least in some cases we still have a possibility result. We derive a sufficient condition for the existence of such a mechanism for the two agents' case, and show an example in which such a condition is satisfied for suitable values of the parameters. Ex-Post IR is a much more restrictive requirement than Interim IR because it requires that, even in the worst case scenario, each agent yields (or at least does not lose) from taking part in the mechanism. Therefore, not only it is consistent with no agents ever feeling regret, but it is clearly the only practicable choice when agents have limited liability or when it is not possible to bind them.

The condition is only sufficient, not necessary. Thus, our result is still partial, but at least shows that it is sometimes possible to design Ex-Post IR institutions without giving up efficiency. A natural development of our work would be to give an answer to the case of n agents. In fact, when there are n agents and types are uniformly distributed over the unit interval, our condition becomes vacuous, in the sense that it is never satisfied. This seems in contrast with the common sense which suggests that, as the market gets larger, it is "easier" to get efficiency.

Appendix

Suppose v_1 and v_2 are independent and uniformly distributed with supports $V_1 = [0, 1]$ and $V_2 = [a, a + 1]$, with $a \in [0, 1)$ and that $u_i(v) = s_i a_i(v) + t_i(v)$, $i = 1, 2$. Let $V = V_1 \times V_2$.

In this environment, the Ex-Post efficient allocation is the one that assigns the whole good to the agent with the highest valuation⁷, i.e.:

$$a_1(v_1, v_2) = \begin{cases} 1 & \text{if } v_1 \geq v_2 \\ 0 & \text{otherwise} \end{cases}, \quad a_2(v_1, v_2) = 1 - a_1(v_1, v_2)$$

and the total welfare associated to this allocation is $g(v) = \max\{v_1, v_2\}$.

The expected welfare to agent 1 is given by:

$$G_1(v_1) = E_{-1}[g(v)] = \begin{cases} \frac{1}{2}v_1^2 - av_1 + \frac{1}{2}(a+1)^2 & \text{if } a < v_1 \leq 1 \\ a + \frac{1}{2} & \text{if } 0 \leq v_1 \leq a \end{cases},$$

⁷We assume that, in case of a tie, the good is assigned to agent 1.

while the expected welfare to agent 2 is given by:

$$G_2(v_2) = E_{-2}[g(v)] = \begin{cases} v_2 & \text{if } 1 < v_2 \leq a + 1 \\ \frac{1}{2}(1 + v_2) & \text{if } a \leq v_2 \leq 1 \end{cases}.$$

We have to determine the following two constants:

$$C_1^* = \inf_{v \in V} \left\{ \frac{1}{2}g(v) + \frac{1}{2} \left(G_1(v_1) - G_2(v_2) \right) - s_1 v_1 \right\},$$

$$C_2^* = \inf_{v \in V} \left\{ \frac{1}{2}g(v) + \frac{1}{2} \left(G_2(v_2) - G_1(v_1) \right) - (1 - s_1)v_2 \right\}.$$

Because the objective function is clearly discontinuous, we divide the admissible set V in five regions and compute the infimum of the objective function within each region. We then pick the minimum of them.

Region 1: $v_1 \in [0, a]$, $v_2 \in (1, a + 1]$.

The objective function for agent 1 is: $\frac{1}{2}v_2 + \frac{1}{2}(a + \frac{1}{2} - v_2) - s_1 v_1$.

The minimum point is $(v_1 = a, v_2 \in V_2)$ and the minimum is: $(\frac{1}{2} - s_1)a + \frac{1}{4}$.

The objective function for agent 2 is: $\frac{1}{2}v_2 + \frac{1}{2}(v_2 - a - \frac{1}{2}) - (1 - s_1)v_2$.

The infimum corresponds to $v_2 \rightarrow 1^+$ and is: $s_1 - \frac{1}{4}(1 + 2a)$.

Region 2: $v_1 \in (a, 1]$, $v_2 \in (1, a + 1]$.

The objective function for agent 1 is: $\frac{1}{2}v_2 + \frac{1}{2}(\frac{1}{2}v_1^2 - av_1 + \frac{1}{2}(a+1)^2 - v_2) - s_1 v_1$.

- If $s_1 \leq \frac{1-a}{2}$, the minimum point is $(v_1 = a + 2s_1, v_2 \in V_2)$ and the minimum is $-s_1^2 + as_1 + \frac{1}{2}a + \frac{1}{4}$.
- If $s_1 > \frac{1-a}{2}$, the minimum point is $(v_1 = 1, v_2 \in V_2)$ and the minimum is $\frac{1}{2} + \frac{1}{4}a^2 - s_1$.

The objective function for agent 2 is: $\frac{1}{2}v_2 + \frac{1}{2}(v_2 - \frac{1}{2}v_1^2 + av_1 - \frac{1}{2}(a+1)^2) - (1 - s_1)v_2$.

The infimum corresponds to $(v_1 = 1, v_2 \rightarrow 1^+)$ and is: $s_1 - \frac{1}{4}(2 + a^2)$.

Region 3: $v_1 \in [0, a]$, $v_2 \in [a, 1]$.

The objective function for agent 1 is: $\frac{1}{2}v_2 + \frac{1}{2}(a + \frac{1}{2} - \frac{1}{2}(1 + v_2^2)) - s_1 v_1$.

The minimum point is $(v_1 = a, v_2 = a)$ and the minimum is $(1 - s_1 - \frac{1}{4})a$.

The objective function for agent 2 is: $\frac{1}{2}v_2 + \frac{1}{2}(\frac{1}{2}(1 + v_2^2) - a - \frac{1}{2}) - (1 - s_1)v_2$.

- If $s_1 \leq \frac{1-a}{2}$, the minimum point is $(v_1 \in V_1, v_2 = 1 - 2s_1)$ and the minimum is $s_1(1 - s_1) - \frac{1}{4}(1 + 2a)$.
- If $s_1 > \frac{1-a}{2}$, the minimum point is $(v_1 \in V_1, v_2 = a)$ and the minimum is $(s_1 + \frac{1}{4}a - 1)a$.

Region 4: $v_1 \in (a, 1]$, $v_2 \in (v_1, 1]$.

The objective function for agent 1 is: $\frac{1}{2}v_2 + \frac{1}{2}(\frac{1}{2}v_1^2 - av_1 + \frac{1}{2}(a+1)^2 - \frac{1}{2}(1 + v_2^2)) - s_1 v_1$.

- If $s_1 \leq \frac{1-a}{2}$, the infimum corresponds to $(v_1 \rightarrow a^+, v_2 \rightarrow a^+)$ and is: $(1 - s_1 - \frac{1}{4}a)a$.
- If $s_1 > \frac{1-a}{2}$, the minimum point is $(v_1 = 1, v_2 = 1)$ and the minimum is: $\frac{1}{2} + \frac{1}{4}a^2 - s_1$.

The objective function for agent 2 is: $\frac{1}{2}v_2 + \frac{1}{2}(\frac{1}{2}(1 + v_2^2) - \frac{1}{2}v_1^2 + av_1 - \frac{1}{2}(a + 1)^2) - (1 - s_1)v_2$.

- If $s_1 \leq \frac{1-a}{2}$, the minimum point is $(v_1 = 1, v_2 = 1)$ and the minimum is: $s_1 - \frac{1}{4}(2 + a^2)$.
- If $s_1 > \frac{1-a}{2}$, the infimum corresponds to $(v_1 \rightarrow a^+, v_2 \rightarrow a^+)$ and is: $(s_1 + \frac{1}{4}a - 1)a$.

Region 5: $v_1 \in (a, 1], v_2 \in [a, v_1]$.

The objective function for agent 1 is: $\frac{1}{2}v_1 + \frac{1}{2}(\frac{1}{2}v_1^2 - av_1 + \frac{1}{2}(a + 1)^2 - \frac{1}{2}(1 + v_2^2)) - s_1v_1$.

- If $s_1 \leq \frac{1-a}{2}$, the infimum corresponds to $(v_1 \rightarrow a^+, v_2 \rightarrow a^+)$ and is: $(1 - s_1 - \frac{1}{4}a)a$.
- If $s_1 > \frac{1-a}{2}$, the minimum point is $(v_1 = 1, v_2 = 1)$ and the minimum is: $\frac{1}{2} + \frac{1}{4}a^2 - s_1$.

The objective function for agent 2 is: $\frac{1}{2}v_1 + \frac{1}{2}(\frac{1}{2}(1 + v_2^2) - \frac{1}{2}v_1^2 + av_1 - \frac{1}{2}(a + 1)^2) - (1 - s_1)v_2$.

- If $s_1 \leq \frac{1-a}{2}$, the minimum point is $(v_1 = 1, v_2 = 1)$ and the minimum is: $s_1 - \frac{1}{4}(2 + a^2)$.
- If $s_1 > \frac{1-a}{2}$, the infimum corresponds to $(v_1 \rightarrow a^+, v_2 \rightarrow a^+)$ and is: $(s_1 + \frac{1}{4}a - 1)a$.

By comparing the infima in the five regions above, we conclude that:

- if $s_1 \leq \frac{1-a}{2}$, $C_1^* = (1 - s_1 - \frac{1}{4}a)a$, $C_2^* = s_1 - \frac{1}{4}(2 + a^2)$;
- if $s_1 > \frac{1-a}{2}$, $C_1^* = \frac{1}{2} + \frac{1}{4}a^2 - s_1$, $C_2^* = (s_1 + \frac{1}{4}a - 1)a$.

Finally, we have that $C_1^* + C_2^* \geq 0$ if and only if $s_1 = \frac{1-a}{2}$ (in this case $C_1^* = \frac{1}{4}a(a + 2)$ and $C_2^* = \frac{1}{4}a(a + 2)$, so that $C_1^* + C_2^* = 0$).

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