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Advertising and production of a seasonal good for a heterogeneous market: from total segment separability to real media*

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Abstract. Market segmentation is a fundamental topic of marketing theory and practice. We bring some market segmentation concepts into the statement of an advertising and production problem for a seasonal product with Nerlove-Arrow's linear goodwill dynamics, along the lines of some analyses concerning the introduction of a new product. We consider two kinds of situations. In the first one, we assume that the advertising process can reach selectively each segment. In the second one, we assume that one advertising medium is available and that it has a known effectiveness segment-spectrum for a non-trivial set of segments. In both cases we study the optimal control problems in which goodwill productivity of advertising is either linear or concave, and good production costs are (convex and) quadratic. We obtain the explicit optimal solutions using the Pontryagin's Maximum Principle conditions.

Keywords: Marketing; Advertising; Optimal Control.

JEL Classification Numbers: M37, M31, C61.

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1 Introduction

Although the infinite horizon formulation is the natural setting for the analysis of some dynamic optimization problems in advertising, as for instance in the recent papers [8], [12], [6], there are special situations for which a finite horizon formulation is obviously needed. Examples of this case are advertising for an event ([7]), introducing a new product ([1]), and advertising for a seasonal product ([2], [3], [4], [5]).

Market segmentation is a fundamental topic of marketing theory and practice. We bring some market segmentation concepts into the statement of an advertising and production problem for a seasonal product with Nerlove-Arrow's linear goodwill dynamics [10], along the lines of some analyses concerning the introduction of a new product. We consider two kinds of situations. In the first one, we assume that the advertising process can reach selectively each segment. In the second one, we assume that one advertising medium is available and that it has a known effectiveness segment-spectrum for a non-trivial set of segments. In the first case each segment may be considered an homogeneous and independent market. The second case is more realistic but more difficult to study from the mathematical point of view. In both cases we study the optimal control problems in which goodwill productivity of advertising is either linear or concave, and good production costs are (convex and) quadratic.

The paper is organized as follows.

In Section 2 we consider a heterogeneous market and we assume that each segment has a stock of goodwill. In this section we describe the goodwill evolution.

In Section 3 we consider a firm which produces, sells and advertises a seasonal product and we assume that the advertising process can reach selectively each segment.

In Section 4 we assume that only one advertising medium is available and that it has a known effectiveness for a set of segments.

2 Market segmentation and goodwill evolution

Let the consumer population be partitioned into groups (segments), each one specified by the value $a \in A$ of a suitable parameter *(segmentation attribute)*.

The set $A = \{0-5, 6-11, 12-19, \ldots\ 65-\omega\}$ (see [9] p. 386) is an example of finite segmentation, where the attribute is *age*, $\omega > 0$ is its maximum observable value and the segmentation parameter represents a subinterval of $[0, \omega]$. Different and typical finite segmentations are obtained using the attribute *gender*, in which case we find $A = \{\text{Female}, \text{Male}\}$, or the couple of attributes *gender–status*, so that $A = \{\text{Female}, \text{Male}\} \times \{\text{Married}, \text{Single}\}$. The latter is an example of so-called multivariate segmentation.

Let $G_a(t)$ represent the stock of goodwill of the product at time t, for the (consumers in the) a segment. We refer to the definition of *goodwill* given by [10] to describe the variable which summarizes the effects of present and past advertising on the demand; the goodwill needs an advertising effort to increase, while it is subject to a spontaneous decay. Here we assume that the goodwill evolution satisfies the set of independent ordinary differential equations

$$\dot{G}_a(t) \;=\; w_a(t) - \delta_a G_a(t,) \;, \qquad a \in A \,, \tag{1}$$

where $\delta_a > 0$ represents the goodwill depreciation rate for the members of the consumer group a and $w_a(t)$ is the effective advertising intensity at time t directed to that same group. For each fixed value of the parameter $a \in A$, i.e. for each segment, the dynamics of the goodwill given by (1) is essentially the same as the one proposed in [10]. Here, consistent with the assumption of distinct goodwill variables for different market segments, we further assume that both the advertising intensity and the goodwill decay parameter may depend on the attribute value a. Assuming that equation (1) describes the dynamics of the system amounts to assuming that the firm may control an advertising process, with such a high segment-resolution, as to be able to reach each segment with the desired intensity. This is the *total segment-resolution* assumption, which, in the extreme, is characteristic of *micromarketing* (see [9], p. 380).

3 Advertising of a seasonal good with total segment–resolution

We consider a firm which produces (or purchases), sells and advertises a seasonal product. The feature of the product being seasonal amounts to assume that production and sales take place in two disjoint and consecutive time intervals. Let $[0,1]$ be the planning interval, that is the seasonality period of the product, and let t_1, $0 \le t_1 \le 1$, be the final time of the production interval, $[0,t_1]$, and the starting time of the sales interval, $[t_1,1]$. We assume that the firm can advertise the product at every time of the seasonality period.

We consider the sales interval $[t_1,1]$ only and we want to determine the optimal advertising policy, in order to maximize the net profit. As the season is a short time horizon, we consider undiscounted costs and revenue.

The goodwill evolution is driven by the media activation intensities $u_a(t) \ge 0$ (the control functions) in such a way that the effective advertising intensity at time t directed to segment a is

$$w_a(t) \;=\; \varphi_a(u_a(t)) \,, \tag{2}$$

where the function $\varphi_a(\cdot)$, the productivity of media activation intensity directed to segment $a \in A$, is a nonnegative, increasing and strictly concave function; we assume it is continuously differentiable, so that $\varphi'_a(\cdot) > 0$ is strictly decreasing and hence invertible. In view of equation (2) the goodwill motion equations (1) look as follows:

$$\dot{G}_a(t) \;=\; \varphi_a(u_a(t)) - \delta_a G_a(t) \;, \qquad a \in A. \tag{3}$$

The values of the goodwill components at the initial time t_1 are known data:

$$G_a(t_1) \;=\; \bar{G}_a \ge 0 \,. \tag{4}$$

Let $q_a > 0$ be the unit cost of activating the advertising medium for the segment a, so that the advertising cost intensity associated with $u_a(t)$ is $q_a u_a(t)$, $a \in A$.

The demand intensity depends linearly on the goodwill function and the sales quantity until time t, $x(t)$, satisfies the differential equation

$$\dot{x}(t) = \sum_{a \in A} \beta_a G_a(t), \quad a \in A, \tag{5}$$

and the initial condition

$$x(t_1) = 0. \tag{6}$$

The parameter $\beta_a \geq 0$ is the marginal demand of goodwill in segment a: its value depends on the dimension of the segment, i.e. number of potential consumers in it, and on the interest of those consumers to the product. The total revenue from sales is $px(1)$, where $p > 0$ is the constant product sales price. Let $c(\cdot)$ be the production cost function of the seasonal good, a nonnegative, increasing and strictly convex function, with $c(0) = 0$; we assume it is continuously differentiable, so that $c'(\cdot) > 0$ is strictly increasing. In a deterministic setting, the firm produces exactly the demanded quantity $x(1)$, so that the total production cost is $c(x(1))$. We observe that if the manufacturer does not advertise towards any segment, $u_a(t) \equiv 0$, $a \in A$, then $G_a(t) = \overline{G}_a e^{-\delta_a t}$ and $x(1) = x_{min}$, where

$$x_{min} = \sum_{a \in A} \frac{\beta_a \overline{G}_a}{\delta_a}(1 - e^{-\delta_a(1-t_1)}).$$

In order to avoid trivial situations we assume that

$$c'(x_{min}) < p, \tag{7}$$

otherwise it would not be convenient to the firm to advertise, nor to produce any quantity of the good greater than x_{min}.

3.1 Advertising and production problem

The advertising and production problem requires to find some media activation intensity functions $u_a(t) \geq 0$, in order to maximize the firm profit

$$J(u) = px(1) - \int_{t_1}^{1} \sum_{a \in A} q_a u_a(t)\, dt - c(x(1)), \tag{8}$$

where u is the vector of components u_a, $a \in A$. The firm profit is the difference between the revenue from sales in $[t_1, 1]$ and the total advertising and production costs. The constraints of the problem are constituted by the motion equations (3) and (5), initial conditions (4) and (6), and the nonnegativity control conditions $u_a(t) \geq 0$, $a \in A$.

Theorem 1 *There exists a unique optimal solution*

$$(u(t), G(t), x(t)) = (\{u_a(t)\}_{a \in A}, \{G_a(t)\}_{a \in A}, x(t)),$$

and the optimal control is

$$u_a(t) \ = \ \psi_a \left(\frac{\delta_a q_a}{\bar{\mu}\beta_a} \left(1 - e^{-\delta_a(1-t)}\right)^{-1} \right) , \qquad a \in A \, , \tag{9}$$

where $\psi_a(\cdot)$ *is the inverse function of the derivative* $\varphi'_a(\cdot)$ *of the advertising productivity at segment* a *and*

$$\bar{\mu} \ = \ p - c'(x(1)) \, . \tag{10}$$

Proof The problem Hamiltonian is

$$\begin{aligned} H(G, u, \lambda, \mu, t) \ = \ & \textstyle\sum_{a \in A} \{-\lambda_0 q_a u_a + \lambda_a \varphi_a(u_a)\} \\ & + \textstyle\sum_{a \in A} \{-\lambda_a \delta_a + \mu \beta_a\} \, G_a \, , \end{aligned} \tag{11}$$

which is a continuously differentiable function of (G, u).

From the Pontryagin Maximum Principle conditions (see [11], p. 85) we obtain that

i) $(\lambda_0, \{\eta_a\}_{a \in A}, \mu_1) \neq 0 \, ;$

ii) $u^*_a(t)$ maximizes

$$-\lambda_0 q_a u_a + \lambda_a(t) \varphi_a(u_a) \, , \quad u_a \geq 0 \, , \ a \in A \, ,$$

which is a concave function of u_a, as far as $\lambda_a(t) \geq 0 \, ;$

iii) virtually everywhere,

$$\begin{aligned} \dot{\lambda}_a(t) \ &= \ \lambda_a(t) \delta_a - \mu(t) \beta_a \, , \\ \dot{\mu}(t) \ &= \ 0 \, ; \end{aligned}$$

iv) $\lambda_0 \in \{0, 1\} \, ;$

v) $\lambda_a(1) = \eta_a = 0, \quad \mu(1) = \lambda_0 \, (p - c'(x(1))) + \mu_1 \, ,$
$\eta_a = 0 \, , \ \mu_1 = 0 \, .$

It is easy to observe that $\lambda_0 = 1$ for all solutions and that there may exist the unique optimal control

$$u^*_a(t) \ = \ \begin{cases} 0 \, , & \lambda_a(t) \leq 0 \, , \\ \psi_a \left(q_a / \lambda_a(t) \right) \, , & \lambda_a(t) > 0 \, , \end{cases} \tag{12}$$

where $\psi_a(\cdot)$ is the inverse function of the derivative $\varphi'_a(\cdot)$. We recall that we have assumed $\varphi_a(\cdot)$ strictly concave, so that $\varphi'_a(\cdot)$ is invertible.

From the adjoint equations (iii) and the transversality conditions (v) we obtain

$$\mu(t) \ \equiv \ \bar{\mu} \, , \tag{13}$$

where $\bar{\mu}$ is given by (10), and

$$\lambda_a(t) \;=\; \frac{\bar{\mu}\beta_a}{\delta_a}\left(1 - e^{-\delta_a(1-t)}\right) . \tag{14}$$

We observe that $x(1)$ is an increasing function of $\bar{\mu}$: in fact as $\bar{\mu}$ increases also $\lambda_a(t)$ increases for all a and $t \leq 1$, because of (14); hence $u_a^*(t)$ increases for all a and $t \leq 1$, because of (9) and the fact that $\psi_a(\cdot)$ is monotonically decreasing; hence $G_a(t)$ increases for all a and $t \leq 1$, because of (3); finally $x(t)$ increases for all $t \leq 1$, because of (5).
As a consequence, $p - c'(x(1))$ is a decreasing function of $\bar{\mu}$. Then, from the non–triviality condition (7), $c'(x_{min}) < p$, it follows that there exists a unique solution $\bar{\mu}$ to the equation (10). Now, since the hypothesis of the Mangasarian sufficiency theorem (see [11], p.105) are satisfied, (9) is the unique optimal control. □

We observe that, if an optimal solution does exist, then the optimal activation level $u_a^*(t)$ decreases as time goes by and at the end of the season $u_a^*(1) = 0$, for all segments $a \in A$.

3.2 Square root advertising productivity and quadratic production costs

Let us consider the special case of square root productivity of media activation levels

$$\varphi_a(u_a) \;=\; \gamma_a\sqrt{u_a}\,, \qquad a \in A\,, \tag{15}$$

where $\gamma_a > 0$, $a \in A$, is a parameter which affects positively the marginal productivity of the medium activation intensity directed to the segment a, and quadratic production cost function

$$c(x) \;=\; c_1 x + \frac{1}{2}c_2 x^2\,, \tag{16}$$

with $c_1 \geq 0$ and $c_2 > 0$.

Now, the inverse of the function φ_a' is $\psi_a(y) = (\gamma_a/2y)^2$, $y > 0$, and the marginal production cost is $c'(x) = c_1 + c_2 x$. In this case the goodwill motion equations are

$$\dot{G}_a\,(t) \;=\; \gamma_a\sqrt{u_a\,(t)} - \delta_a G_a\,(t)\;, \qquad a \in A\,. \tag{17}$$

We obtain that there exists the unique set of optimal advertising activation levels

$$u_a^*(t) \;=\; \frac{\bar{\mu}^2}{4}\left(\frac{\beta_a\gamma_a}{q_a\delta_a}\right)^2\left(1 - e^{-\delta_a(1-t)}\right)^2 , \tag{18}$$

where the parameter $\bar{\mu}$ is determined by the transversality condition (10), which is

$$\bar{\mu} \;=\; p - c_1 - c_2 x(1)\,. \tag{19}$$

3.3 Limit case: linear advertising productivity

Let us consider the limit case of linear productivity of media activation levels with bounded domains, i.e.

$$\varphi_a(u_a) \; = \; u_a \,, \tag{20}$$

so that the goodwill motion equations are

$$\dot{G}_a\,(t) \; = \; u_a\,(t) - \delta_a G_a\,(t) \;, \qquad a \in A \,, \tag{21}$$

and the media activation levels are constrained by

$$u_a(t) \; \in \; [0, \, \bar{u}_a] \,, \qquad a \in A \,, \tag{22}$$

where $\bar{u}_a > 0$, $a \in A$, is the maximum medium activation intensity in the segment a. Again we assume quadratic production cost functions as defined in (16).

With the choice (20), the relevant assumption of strict concavity of $\varphi_a(\cdot)$ is not satisfied any more. Hence Theorem 3.1 does not apply here. Nevertheless, the Pontryagin Maximum principle conditions are the same as for the general advertising and production problem, with condition (ii) substituted by:

ii′) $u_a^*(t)$ maximizes

$$[-\lambda_0 q_a + \lambda_a(t)]\, u_a \,, \qquad u_a \in [0, \, \bar{u}_a] \,, \; a \in A \,. \tag{23}$$

We obtain that there exists the unique optimal control

$$u_a^*(t) \; = \; \begin{cases} 0 \,, & \lambda_a(t) < q_a \,, \\ \\ \bar{u}_a \,, & \lambda_a(t) > q_a \,, \end{cases} \tag{24}$$

where the adjoint functions are those of equation (14) and the parameter $\bar{\mu}$ is determined by the special transversality condition (19). More explicitly, if $\frac{\bar{\mu}\beta_a}{\delta_a}(1 - e^{-\delta_a(1-t_1)}) > q_a$, the optimal media activation levels are

$$u_a^*(t) \; = \; \begin{cases} \bar{u}_a, & t \in [t_1, t_a^*] \,, \\ \\ 0, & t \in (t_a^*, 1] \,, \end{cases} \tag{25}$$

where $t_a^* \in (t_1, 1)$ is

$$t_a^* \; = \; 1 + \frac{1}{\delta_a} \ln\left(1 - \frac{q_a \delta_a}{\bar{\mu}\beta_a}\right) . \tag{26}$$

We observe that $t_a^* < 1$, so that it is not optimal to advertise until the end of the sale period toward any segment. Moreover, the firm will advertise longer towards a segment a as the segment marginal demand β_a is larger and as the advertising cost parameter q_a is smaller: an intuitive result qualitatively. On the contrary, if $\frac{\bar{\mu}\beta_a}{\delta_a}(1 - e^{-\delta_a(1-t_1)}) \leq q_a$, the optimal activation level of the medium a is $u_a^*(t) \equiv 0$.

4 Problem with partial segment-resolution of advertising

Let us consider the situation in which the decision maker has to use an advertising medium which reaches several segments with variable effectiveness, instead of using a set of segment-specific media. Let $u(t) \geq 0$ be the activation level of the advertising medium and let the effective advertising intensity at time t directed to segment a be

$$w_a(t) \ = \ \varphi_a(u(t)), \qquad a \in A. \tag{27}$$

In particular we assume here that

$$\varphi_a(u) = \gamma_a \varphi(u), \tag{28}$$

for some segment specific parameters $\gamma_a > 0$, $a \in A$, and a function $\varphi(\cdot)$, so that the goodwill motion equations (1) look as follows:

$$\dot{G}_a(t) \ = \ \gamma_a \varphi(u(t)) - \delta_a G_a(t), \qquad a \in A. \tag{29}$$

The function $\varphi(\cdot)$, the productivity of the medium intensity, is a nonnegative, increasing and strictly concave function; furthermore it is continuously differentiable, so that $\varphi'(\cdot) > 0$ is strictly decreasing and hence invertible. Let $\gamma_a \geq 0$, $a \in A$ and $\sum_{a \in A} \gamma_a = 1$. We call $(\gamma_a)_{a \in A}$ the medium *(segment-)spectrum*. Its components, γ_a, $a \in A$, provide the different relative effectiveness of the advertising medium on the market segments. Finally, let $q > 0$ be the unit cost of activating the advertising medium.

The advertising and production problem requires to find a medium activation intensity function $u(t) \geq 0$, in order to maximize the firm profit given by the functional

$$J(u) \ = \ px(1) - q \int_{t_1}^{1} u(t)\, dt - c(x(1)), \tag{30}$$

under the conditions represented by the goodwill and sales motion equations (29) and (5) and the initial conditions (4) and (6).

Theorem 2 *There exists a unique optimal solution*

$$(u(t), G(t), x(t)) = (u(t), \{G_a(t)\}_{a \in A}, x(t)),$$

and the optimal control is

$$u(t) \ = \ \psi\left(\frac{q}{\bar{\mu}} \left[\sum_{a \in A} \frac{\gamma_a \delta_a}{\beta_a} \left(1 - e^{-\delta_a(1-t)}\right) \right]^{-1} \right), \tag{31}$$

where $\psi(\cdot)$ *is the inverse function of the derivative* $\varphi'(\cdot)$ *of the medium productivity and*

$$\bar{\mu} \ = \ p - c'(x(1)). \tag{32}$$

Proof The problem Hamiltonian is

$$H(G,u,\lambda,\mu,t) \;=\; -\lambda_0 qu + \varphi(u)\textstyle\sum_{a\in A}\lambda_a\gamma_a \\ \quad + \textstyle\sum_{a\in A}\{-\lambda_a\delta_a + \mu\beta_a\}\,G_a\,, \tag{33}$$

which is a continuously differentiable function of (G,u).
From the Pontryagin Maximum Principle conditions (see [11], p.85) we obtain the same conditions (i), (iii), (iv), (v) as in the analysis of the total segment resolution problem in Section 3 and the maximum condition

ii) $u^*(t)$ maximizes

$$-\lambda_0 qu + \varphi(u)\sum_{a\in A}\gamma_a\lambda_a(t)\,,$$

which is a concave function of u, as far as $\sum_{a\in A}\gamma_a\lambda_a(t) \geq 0$.

We observe that $\lambda_0 = 1$ for all solutions and that there may exist the unique optimal control

$$u^*(t) \;=\; \begin{cases} 0\,, & \sum_{a\in A}\gamma_a\lambda_a(t) \leq 0\,, \\ \psi\left(q/\sum_{a\in A}\gamma_a\lambda_a(t)\right)\,, & \sum_{a\in A}\gamma_a\lambda_a(t) > 0\,, \end{cases} \tag{34}$$

where $\psi(\cdot)$ is the inverse function of the derivative $\varphi'(\cdot)$.

The adjoint variables $\lambda_a(t)$, $a \in A$, and $\mu(t)$, are the same as in (14) and (10), because are determined by the same equations (iii) and transversality conditions (v). So the conclusions are the same as in in the analysis of the total segment resolution problem in Section 3. □

4.1 Square root advertising productivity and quadratic production costs

Let us consider the special case of square root productivity of the medium activation level

$$\varphi(u) \;=\; \sqrt{u}\,, \tag{35}$$

and quadratic production cost function (16).

The inverse of φ' is $\psi(y) = (2y)^{-2}$, $y > 0$, and the marginal production cost is $c'(x) = c_1 + c_2 x$. The goodwill motion equations are

$$\dot{G}_a(t) \;=\; \gamma_a\sqrt{u(t)} - \delta_a G_a(t)\,, \qquad a \in A\,. \tag{36}$$

The unique optimal advertising activation level is

$$u^*(t) \;=\; \frac{\bar{\mu}^2}{4q^2}\left[\sum_{a\in A}\frac{\beta_a\gamma_a}{\delta_a}\left(1 - e^{-\delta_a(1-t)}\right)\right]^2\,, \tag{37}$$

where the parameter $\bar{\mu}$ is determined by the transversality condition (10).

4.2 Limit case: linear advertising productivity

Let us consider the limit case of linear productivity of the medium activation level with bounded domain, i.e.

$$\varphi(u) \;=\; u\,, \tag{38}$$

so that the goodwill motion equations are

$$\dot{G}_a\,(t) \;=\; \gamma_a u\,(t) - \delta_a G_a\,(t)\;, \qquad a \in A\,, \tag{39}$$

and the medium activation level is constrained by

$$u(t) \;\in\; [0,\, \bar{u}]\,, \tag{40}$$

where $\bar{u} > 0$. Again we assume quadratic production cost functions as defined in (16).

We observe, as done in Section 4.2, that the function $\varphi_a(\cdot)$ is not strictly concave, as required by Theorem 4. The Pontryagin Maximum principle conditions are the same as those analysed in the proof of Theorem 4, with condition (ii) substituted by:

ii′) $u^*(t)$ maximizes

$$\left[-\lambda_0 q + \sum_{a\in A} \gamma_a \lambda_a(t)\right] u_a\,, \qquad u \in [0,\, \bar{u}]\,. \tag{41}$$

We obtain that there exists the unique optimal control

$$u^*(t) \;=\; \begin{cases} 0\,, & \sum_{a\in A} \gamma_a \lambda_a(t) < q\,, \\[2ex] \bar{u}\,, & \sum_{a\in A} \gamma_a \lambda_a(t) > q\,, \end{cases} \tag{42}$$

where the adjoint functions are those of equation (14) and the parameter $\bar{\mu}$ is determined by the special transversality condition (19). More explicitly, the optimal medium activation level is

$$u^*(t) \;=\; \begin{cases} \bar{u}, & t \in [t_1, t^*]\,, \\[2ex] 0, & t \in (t^*, 1]\,, \end{cases} \tag{43}$$

where $t^* \in [t_1, 1]$ is the unique solution to the equation

$$\sum_{a\in A} \frac{\gamma_a \beta_a}{\delta_a} \left(1 - e^{\delta_a(1-t)}\right) \;=\; \frac{q}{\bar{\mu}}\,. \tag{44}$$

Again we observe that $t^* < 1$, so that it is not optimal to advertise until the end of the sale period.

5 Conclusion

In the paper we have brought some market segmentation concepts into the statement of an advertising and production problem for a seasonal product with Nerlove-Arrow's linear goodwill dynamics, along the lines of some analyses concerning the introduction of a new product. We have considered two kinds of situations. In the first one, the advertising process can reach selectively each segment. In the second one, only one advertising medium is available and it has a known effectiveness segment-spectrum over the segment set. In both cases we have studied the optimal control problems in which goodwill productivity of advertising is either linear or concave, and good production costs are (convex and) quadratic and we have obtained the explicit optimal solutions, using the Pontryagin's Maximum Principle conditions.

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