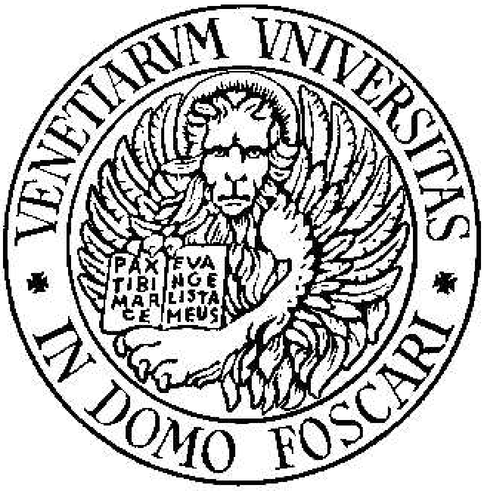
Department of Applied Mathematics, University of Venice

QUADERNI DI DIDATTICA



Paolo Pianca

The St. Petersburg Paradox: Historical Exposition, an Application to Growth Stocks and Some Simulation Approaches

Quaderno di Didattica n. 24/2007
Settembre 2007

The St. Petersburg Paradox: Historical Exposition, an Application to Growth Stocks and Some Simulation Approaches

PAOLO PIANCA

Department of Applied Mathematics
University Ca' Foscari Venice

Abstract. The paradox of the St. Petersburg game is one of the oldest classical problems in probability theory. The payer's payoff is doubled as long as the coin he flips shows head; therefore the game is characterized by an infinite expectation value. This shows that no finite amount of money can be a fair entrance fee.

The present contribution summarizes some attempts to resolve the paradox and analyzes a fascinating paper in which the valuation of growth stocks is relate to the St. Petersburg game. The conclusion is that the run-up in high-tech stock price in the late of 1990's and the subsequent declines could be avoided by an analysis and an application of St. Petersburg game.

Last section concerns on some approaches to simulate the St. Petersburg paradox. The main result is that the theoretical and simulated outcomes are in agreement only if the game is played an infinite number of times, a practical impossibility.

Keywords: St. Petersburg game; utility function; fair game of chance; valuation of growth stocks; Alan Greenspan; simulation techniques.

JEL Classification Numbers: G11, C44, C63.

MathSci Classification Numbers: 60E15.

Correspondence to:

Paolo PIANCA
Dept. of Applied Mathematics, University of Venice
Dorsoduro 3825/e
30123 Venezia, Italy
Phone: [+39] (041)-234-6915
Fax: [+39] (041)-522-1756
e-mail: pianca@unive.it

1 Introduction

In the early days of the calculus of probability it was taken for granted that the value and hence the "fair price" of a gamble, was the mathematical expectation of the game. Thus this price would be

$$\mathbb{E}[X] = \int_0^\infty x\, dF(x) \tag{1}$$

if X represents the gains of the gamble having cumulative probability distribution function F. Applied to insurance this means that the fair premium p for a risk described by the non-negative variable X would be

$$p = \int_0^\infty x\, dF(x). \tag{2}$$

Recall that $\mathbb{E}[X]$ represents the long-term average value of X after many repetitions of the game. Intuitively, the game is fair if, after a large number of repetitions of the game, neither the first player nor the second player has an advantage over each other.

As Feller [9] observed in an extensive discussion, this definition of a fair game, as a rule, applied only if the variance of X is finite.

Daniel Bernoulli [2] published an example, originally communicated on September 9, 1713 to Reimond de Montmort by his cousin Nicolaus Bernoulli[*], where the above method does not work simply because the integral above does not converge. Such an example is well known as the *St. Petersburg Paradox* and deals with a game where the player Peter tosses a fair coin[†] and continues to do so until the coin falls on head. He agrees to pay to the player Paul two ducats if a head appears on the first toss, four ducats if the first head appears on second toss, eight ducats if the first head appears on third toss, etc. How much should Peter charge Paul as an entrance fee to this game so that the game will be fair? In this game the potential payoff doubles after each toss. We will refer to games of this type as St. Petersburg

---

[*] The originator of the problem was Nikolaus Bernoulli (1687-1759), who ultimately became a professor of jurisprudence in Basel but had been well trained in mathematics as a young man. As well known, the Bernoulli were a renowned family of Swiss mathematicians, several members of whom studied probability theory; particularly notable for his work in that field was James Bernoulli, the father of the first law of large numbers.

[†] In a letter to Nicolaus Bernoulli written on May 21,1728 and mailed from London to Basel, Gabriel Cramer, father of Cramer's rule for solving systems of linear equations, rephrased the problem from dice to coins "for simplicity". In this letter Cramer proposed to Nikolaus a solution. Nikolaus was dissatisfied with this and in 1728 began a correspondence on the problem with his cousin Daniel, then resident member of the Imperial Academy at St. Petersburg.

games. Note that in Daniel Bernoulli's original statement of the paradox, the payoff for k tosses is 2^{k-1} ducats, but we have chosen to work throughout with a payoff of 2^k ducats in order to simplify calculation of expectations.

In order to determine the amount Peter should charge Paul as an entrance fee so that the game will be fair, we need to calculate Paul's payoff. Surprisingly, the game cannot be faire, no matter how large the entrance fee is.

To show this, we calculate Paul's expected payoff as follows. Let k any positive integer; then 2^{-k} is the probability that the game ends at the kth toss, at which time Peter will pay Paul 2^k ducats. Let Y denote Peter's payout; then Y is a random variable with possible values $2, 4, 8, \ldots$, and

$$\mathbb{P}(Y = 2^k) = 2^{-k} \qquad\qquad k \in \mathbb{N} = \{1, 2, \ldots\}. \tag{3}$$

Equivalently, $\log_2 Y$ has a geometric distribution with probability of success $1/2$. Therefore the expected value of Y is

$$\mathbb{E}[Y] = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} + \ldots = 1 + 1 + 1 + \ldots, \tag{4}$$

which shows that Y has infinite expectation. This proves that no finite amount of money can be a fair entrance fee. In short, Paul should be willing to pay an infinite price to enter this game. However, this is a requirement to which almost no rational person would agree.

Nicolaus argued and Daniel cites him, that no rational person would be willing to pay an arbitrary large amount for the right to participate in this gamble. He is in fact more explicit about it, and writes that "there should be no sensible man who would not be willing to sell his right to this gain for 20 ducats". In short, although the calculation of Paul's expectation is mathematically correct, the conclusion must be regarded as paradoxical. Therefore several probabilists suggested modifications to the game so as to obtain an acceptable solution.

2 Some attempts to resolve the paradox

The many attempts to resolve the paradox fall mostly into two broad groups: those denying the basic assumptions of the game as unrealistic, and those arguing from additional assumptions that the value of the game to Paul is less than its mathematical expectation.

A number of writers chose to accept the assumption of an indefinitely prolonged game and to direct their attention toward ascertaining the value of such a game to Paul.

As an alternative to the expected gain, Daniel Bernoulli suggested that a person would assign the "moral value" of $\log x$ to a gain of x. The value of gamble was then suggested to be the "moral expectation"

$$\mathbb{E}[\log X] = \sum_{n=1}^{\infty} \log 2^n \frac{1}{2^n} = 2 \log 2, \tag{5}$$

a finite number which corresponds to a monetary amount of four ducats. Daniel Bernoulli regarded the "paradox" as resolved, and assigned this finite number as the "price" of the lottery. This is of course a very "ad hoc" solution which would not help if the gain was changed to 2^{2^k} instead of 2^k.

For another modification to the game let us suppose, like Buffon (well known for "Buffon's needle") and Cramer, that utility of money ceases to increase beyond a certain amount, say one million ducats. Or equivalently, let us make the natural assumption that Peter has limited resources, in which case he necessarily must place a limit on the size of his payouts. Bearing in mind that $2^{19} < 10^6 < 2^{20}$, it follows that Peter payout in ducats is

$$Y = \begin{cases} 2^k & \text{if} \quad 1 \le k \le 19 \\ 10^6 & \text{if} \quad k \ge 20. \end{cases} \tag{6}$$

Further, the corresponding probability are given by

$$\mathbb{P}(Y = y) = \begin{cases} \frac{1}{y} & \text{if} \quad y = 2, 4, 8, \ldots, 2^{19} \\ 2^{-20} + 2^{-21} + \ldots & \text{if} \quad y = 10^6. \end{cases} \tag{7}$$

The expected value of Peter's payouts is then seen to be

$$\begin{aligned} \mathbb{E}[Y] &= \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \ldots \frac{1}{2^{19}} \cdot 2^{19} + \left(\frac{1}{2^{20} + 2^{21} + \cdots} \right) \cdot 10^6 && (8) \\ &= 19 + \frac{10^6}{2^{19}} = 20.9073\ldots < 21. && (9) \end{aligned}$$

Therefore, if Paul is required to pay an entrance fee of twenty-one ducats, then the game becomes slightly favorevole to Peter.

Others have noted that the St. Petersburg game seems unrealistic, given that natural limits exist on the time and financial resources of Peter and Paul. From the point of view of those critics, the St. Petersburg problem is not even a paradox. As D. Durand (see [8], p. 352) eloquently puts it, "Peter and Paul are mortal; so after a misspent youth, a dissipated middle age, and a dissolute dotage, one of them will die, and the game will cease, heads or no heads. Or again, Peter's solvency is open question, for the stakes advance at an alarming rate.... Even if Peter and Paul agree to cease after 100 tosses, the stake, though finite, staggers the imagination".

Another extensive assessment which concludes that the St. Petersburg game is not a paradox is provided by Samuelson in [13] where a modern exposition of the game is presented, with particular emphasis on the economic implications.

An interesting variant of Bernoulli's approach was proposed by W. A. Whitworth [18]. Whitworth's resolution is independent of arbitrary devices for measuring the utility of money and it considers the possibility of gambler's ruin, a natural concern for all gamblers with finite resources. Whitworth assumed that prudent gamblers would place at risk a fixed percentage, rather than a fixed amount, of their funds, and he constructed a procedure for analyzing ventures that involve risk of ruin. It follows from Whitworth's study that Paul's entrance fee should depend on the size of his funds.

W. Feller [9] proposed a different method to determine entrance fees which would make fair the St. Petersburg game. Suppose Paul chooses to play the game repeatedly. After n games have been played, let R_n denote the total entrance fees and let S_n denote Paul's accumulated receipts. Let us call the game asymptotically fair if the ratio S_n/R_n converges to 1 in probability as n tends to infinity, i.e., if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{R_n} - 1\right| < \varepsilon\right) \to 1. \tag{10}$$

Feller proved that the St. Petersburg game becomes asymptotically fair if $R_n = n\log_2 n$. From the calculation of Paul's expected payoff, we see also that the game cannot be asymptotically fair if there is a fixed entrance fee per game, i.e. if $R_n = cn$ where c is a finite constant. However, if entrance fee per game may depend on the number of games already played then, according to Feller's theorem, the St. Petersburg paradox is resolved (see [10], pp. 235–237).

Following on the idea of varying entrance fees as initiated by Feller in [9], a deterministic sequence of entrance fees for the St. Petersburg game,

$$2 \quad 4 \quad 2 \quad 8 \quad 2 \quad 4 \quad 2 \quad 16 \quad 2 \quad 4 \quad 2 \quad 8 \quad 2 \quad 4 \quad 2 \quad 32 \quad 2 \quad 4 \quad 2\ldots,$$

was given in [14] by Steinhaus. To construct this sequence, place twos in alternating empty places, then fill every second empty place by a four, next fill every second remaining empty space by an eight, etc. Denote by $a_1, a_2, \ldots$ the members of this sequence, and let a_n the entrance fee at the nth repetition of a St. Petersburg game. Steinhaus proved that, with probability one, the sequence of actual gains will have the same distribution function as the sequence $\{a_a\}$.

More recently, Csörgö and Simons [7] provided an extensive discussion of Steinhaus sequence of entrance fees, and there now is an extensive literature on asymptotic theory for St. Petersburg games. We refer to Vardi [16], Berkes et al. [1], S. Csörgö [6], Blavatskyy [3], and Rieger and Wang [11] for the latest developments in order to resolve the paradox and for further references.

3 The St.Petersburg game and the valuation of growth stocks

A remarkable financial application of the St. Petersburg game, proposed by Durant in [8] and then reproposed with more details by Székely and Richards in [15], relates to the valuation of the common stocks of "growth" companies. Here a "growth" company is one whose revenues are growing significantly faster than the overall economy; the stocks of this company are commonly referred to as "growth" stocks. We shall review this application in some detail because of its applicability to financial trends in the late 1990's and early 2000's.

As background, we recall that during the late 1990's, an unprecedented increases had occurred over the previous three years in the share prices of "growth" common companies. These increases in stock prices sparked a lively debate over whether investor were judicious to purchase these shares, or stupid not to purchase greater amounts before prices increased even further. On the one hand there were those, like Alan Greenspan, chairman of the board of governors of Federal Reserve System, who raised concerns about inflationary pressures and dramatic instabilities in world markets resulting from increased stock prices. On the other hand there were believers, including many mutual funds, which had purchased substantial numbers of shares of high-tech companies.

Returning to the paradox, recall that the St. Petersburg game entails the payment of amounts $2, 4, 8, 16, \ldots$, with corresponding probability

$$1/2, 1/4, 1/8, 1/16 \ldots .$$

Following [8] and [15], we consider a modified St. Petersburg game in which Peter is a growth company and Paul is a perspective purchaser of Peter's stock. Moreover, we suppose that the probability of tossing heads is $i/(1+i)$ ($i > 0$); then the probability of tails is $1/(1+i)$. Next, suppose that the corresponding payoffs are a series of increasing payment in which Peter pays Paul D ducats if the first toss is a tail, $D(1+g)$ ducats if the second toss is a tail, $D(1+g)^2$ ducats if the third toss results in a tail, etc., and this continues until the toss results in heads, at which point the game ends. If k tosses are needed for the game to end then the total payoff to Paul is

$$\sum_{j=0}^{k-2} D(1+g)^j = \frac{D[(1+g)^{k-1}-1]}{g}. \tag{11}$$

Because heads and tails appear with probability $i/(1+i)$ and $1/(1+i)$, respectively, then the payment (11) occurs with probability $i/(1+i)^k$. As Durand [8] observed, Paul's expected payoff is given by the double summation

$$\sum_{k=1}^{\infty} \frac{i}{(1+i)^k} \sum_{j=0}^{k-2} D(1+g)^j. \tag{12}$$

This double sum can be evalueted by substituting for the inner sum the closed form expression (11). Alternatively, by reversing the order of summation and evaluating the resulting inner sum, we find that Paul's expected payoff is

$$\sum_{k=1}^{\infty} \frac{D(1+g)^{k-1}}{(1+i)^k} = \begin{cases} D/(i-g) & \text{if} \quad g < i \\ \infty & \text{if} \quad g \geq i. \end{cases} \tag{13}$$

In summery, if $g < i$ then Paul's expected payoff is $D/(i-g)$; while, if $g \geq i$ then Paul's payoff is infinite, in which case a wise Paul would decline to pay the corresponding entrance fee.

In a contest of appraising the values of financial securities, the parameter i represents an effective rate of return. In the appraisal of fair value for a company's share, g represents the growth rate of the company as measured by the compound increase in revenue per share. A widely-accepted approach for calculating a fair value of Peter's stock is to discount all future dividends in perpetuity. Here, a fair value for Peter's stock is estimated by the present value of all future dividends. Denote by E_n and B_n the size of Peter's earnings per share (i.e. profits) and book value (net asset value) per share, respectively, in year n. Further, denote by D_n the total of Peter's paid-out dividends per share in year n. A moment's reflection makes clear that changes in book value from year-to-year are equal to the difference between earning and dividends paid, that is,

$$B_{n+1} - B_n = E_n - D_n \qquad \forall\ n \geq 1. \tag{14}$$

In estimating fair valuation of Peter's stock, it is common practice for Paul to assume that the ratios

$$r = \frac{E_n}{B_n} \qquad \text{and} \qquad p = \frac{D_n}{E_n}$$

are independent of n. This assumption implies that the change in book value, from year n to year $n+1$, is a constant multiple of E_n

$$B_{n+1} - B_n = E_n - D_n = (1-p)E_n = (1-p)r\,B_n. \tag{15}$$

Therefore Peter's dividends, book value and earnings are all growing at a constant rate, $g = (1-r)p$. In this context the relation (13) represents a perpetual sum of dividends payments, starting at D ducats, growing at a constant rate g, and discounted at rate i in perpetuity. If $i > g$ then the sum (13) converges to

$$\frac{D_1}{i-g} = \frac{p\,E_1}{i-g} = \frac{B_1 rp}{i-g}, \tag{16}$$

which represent an estimate of fair value for the share of Peter's stock.

If $i \leq g$ then the series (13) diverges to infinity, and in this case we have a form of the St. Petersburg paradox in which the practice of discounting future dividends at a uniform rate in perpetuity leads to a paradoxical result.

In summary, the St. Petersburg game can explain some of the irrational increases in the prices of high-tech stock in the late 1990's. During that period, the Federal Reserve discount rate was near to a historical low. Moreover, buyers of growth company stocks assumed that the growth rate g would remain high in perpetuity. The outcome was that $i < g$; indeed, even more extreme was that for many high-tech companies $i/g \simeq 0$. By discounting earnings and dividends in perpetuity, any use of the valuation formula (13) leads to estimated valuations of tech-stocks prices at levels as high as any market price must be considered a profitable price.

Having applied the previous valuation formula to obtain exorbitant estimated valuations for many high-tech growth stocks, stock purchasers bought avidly, thereby forcing prices to extreme level. As well known, by late 2000, stock prices underwent the prolonged contractions predicted by Greenspan, with subsequent devastating losses to many purchasers.

4 Simulating the St. Petersburg game

In the age before the computers Buffon [4] seems to be the first who, , really tried to play a large number M of plays and calculate the empirical mean

$$\frac{S_M}{M} = \frac{1}{M}\sum_{k=1}^{M} Y_k \tag{17}$$

where Y_k is the payoff in play no. k. He let a child play the game 2,048 times and derived the empirical mean. The results obtained are summarized in Table 1.

Tosses(k)	Sequence	Observ. Freq.	Theor. Freq.	Payoff (2^k)
1	H	1061	1024	2
2	TH	494	512	4
3	T^2H	232	256	8
4	T^3H	137	128	16
5	T^4H	56	64	32
6	T^5H	29	32	64
7	T^6H	25	16	128
8	T^7H	8	8	256
9	T^8H	6	4	512

Table 1: Buffon's Simulation of the St. Petersburg Game.

A pedagogical Monte Carlo simulation of the St. Petersburg game is presented by Ceasar in [5]. The empirical analysis carried out refers to the utility functions proposed by Bernoulli ($B(x) = 5 \log x$) and Cramer ($C(x) = \sqrt{x}$). The respective expected utilities of the St. Petersburg game are

$$\mathbb{E}[B(k)] = \sum_{k=1}^{\infty} \frac{5 \log 2^k}{2^k} \qquad \mathbb{E}[C(k)] = \sum_{k=1}^{\infty} \frac{\sqrt{2^k}}{2^k}. \tag{18}$$

Therefore, to participate in this game it should be willing to pay up to $e^{6.931/5} = 4$ ducats (function B) or up to $2.414^2 = 5.83$ ducats (function C). The focus of Ceasar's simulation is to provide some insight on how the cumulative utility averages tend to their respective expected utilities. The simulation points out that the convergence occurs fairly rapidly after 2,000 trials.

In 1992 Russon and Chang [12] simulated St. Petersburg games and find that their results are not consistent with their theoretical prediction. They introduce the concept of practical expected value for a game and in particular for the St. Petersburg game. This concept is based on observation that in a sequence of random events like coin flippings, large number of successive identical outcomes (e.g. 20 heads in a row) do not usually happens. Russon and Chang affirm that although the theoretical probabilities of such rare outcomes are nonzero, their practical probabilities are zero simply because in reality we are not practically able to repeat such trials sufficiently large number of times to realize them. Their paper quotes the results of a large simulation are; the game is slightly modified by limiting the maximum number of tosses. For each specified maximum number of tosses (from 1 to 40) the coin is flipped until the heads appears or the maximum number of tosses is reached, which comes first, and the appropriate payoff is recorded. The coin flipping is done 5,000 times for each maximum toss and the process is repeated five times to generate five samples of 5,000 tosses for each specified maximum toss. The simulated expected values for the combined samples of 25,000 tosses are virtually identical to the theoretical values for maximum tosses of 1 to 16. However, in 20 out the remaining 24 cases (from 17 to 40 tosses), the simulation averages are essentially flat and fall below the theoretical averages generating negative differentials. Some statistical analysis confirms this stabilizing pattern of simulated expected values and the authors conjecture that the stability will continue to exist as the number of tosses is increased to infinity. Moreover, since Russon and Chang believe that large simulation carried out is a device that reasonably depicts the game player's subjective perception about the outcome of the game, they argue that overall mean average payoff, equal to 8.90, should specify the amount of money risk-neutral individuals would be willing to pay to play the game and suggest that risk aversion can be deduced if a person pays a amount less that this practical expected value. Their conclusions are that although the theoretical probabilities exist for large numbers of successive trials, such outcomes simply do no occur in reality and that their research suggests the value 8.90 as a practical expected fee of the game.

Recently, the problem of simulating the St. Petersburg game has been reviewed by Vivian in [17] where it is proved that the consistency does indeed exist between theory and simulated results when the theoretical predictions are derived in terms of the methodology originally suggested by Bernoulli.

Vivian observes that if the game is played M times, $M/2$ games terminate with the first flip of the coin, $M/4$ with the second flip, $M/8$ with the third flip, and so on. Clearly for a finite number of games, the total number of games produces a short series of terms. For example, if $M = 2^{12}$ (4,096) games are played and using the a priori probability of half for a head or tail appearing with each flip of the coin, the number of games expected to terminare after each flip is

$$2,048 (= 2^{11}),\ 1,024,\ 512,\ 256,\ 128,\ 64,\ 32,\ 16,\ 8,\ 4,\ 2,\ 1$$

and the aforementioned series consists of 12 terms, starting with the value of 2^{11} and ending with a value of 1. Moreover, if the terms of the series are summated the total is $2^{12} - 1 = 4,095 = M - 1$. Thus theoretically, using a priori probability, all games except one terminate within the first k terms of the series. Theoretically therefore, none of these $M - 1$ games continues to infinity The total series consists of $k - 1$ terms. Only a single game continues beyond the kth term. This game could of course terminate at any term

Term		T1	T2	T3	T4	T5	T6	T7	T8	T8	T10	T11	T12	T13	T14	T15
Payout		1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384
k	M															
0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	**1**	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	4	1	**0**	3	0	0	0	0	0	0	0	0	0	0	0	0
3	8	6	1	**0**	0	1	0	0	0	0	0	0	0	0	0	0
4	16	9	2	4	**1**	0	0	0	0	0	0	0	0	0	0	0
5	32	16	8	3	3	**1**	1	0	0	0	0	0	0	0	0	0
6	64	32	14	7	3	5	**2**	1	0	0	0	0	0	0	0	0
7	128	63	56	18	9	3	3	**2**	0	0	0	0	0	0	0	0
8	256	99	90	39	16	5	5	1	**1**	0	0	0	0	0	0	0
9	512	270	96	83	31	15	8	8	0	**1**	0	0	0	0	0	0
10	1024	544	233	129	56	33	11	9	3	2	**4**	0	0	0	0	0
11	2048	1040	495	242	115	76	44	24	4	3	4	**0**	0	0	1	0
12	4096	2051	993	518	264	125	74	33	17	9	6	3	**1**	1	1	0
13	8192	4026	2083	1037	328	261	123	71	32	14	13	3	0	**1**	0	0
14	16384	8198	4055	2071	1008	510	267	135	73	28	16	4	2	0	**0**	0
15	32768	16480	8198	4033	2079	1067	488	259	128	69	28	14	8	4	1	**1**
16	65536	32716	16474	8092	4034	2091	1048	511	299	139	68	39	12	5	2	4
17	131072	65637	32712	16502	7993	4128	2089	990	491	267	131	67	31	17	7	4
18	262144	130491	65756	33096	16436	8156	4130	2061	1044	491	257	109	58	29	17	9
19	524288	261835	131346	65356	32866	16459	8187	4140	2131	895	494	247	123	51	40	15
20	1048576	525514	261500	130798	65274	32614	16431	8232	4088	2010	1050	524	261	132	71	40

Table 2: Result of simulating St. Petersburg paradox versus theoretical predictions. The bold numbers indicate the position in the series where theoretically only one game is expected to exist. Furthermore, only one game is expected to appear beyond that position.

Term		T16	T17	T18	T19	T20	T21	T22	T23	T24	EMV	EMV			
Payout		32768	65536	131072	252144	524288	1048576	2097132	4194304	8388608	Empirical	A priori			
k	M										Simulation	50%	75%	87.5%	93.75%
0		0		0	0	0	0	0	0	0	2.00	1.0	2.0	4.0	8.0
1	2	0	0	0	0	0	0	0	0	0	1.50	1.5	2.5	4.5	8.5
2	4	0	0	0	0	0	0	0	0	0	3.25	2.0	3.0	5.0	9.0
3	8	0	0	0	0	0	0	0	0	0	3.00	2.5	3.5	5.5	9.5
4	16	0	0	0	0	0	0	0	0	0	2.31	3.0	4.0	6.0	10.0
5	32	0	0	0	0	0	0	0	0	0	3.63	3.5	4.5	6.5	10.5
6	64	0	0	0	0	0	0	0	0	0	5.00	4.0	5.0	7.0	11.0
7	128	0	0	0	0	0	0	0	0	0	4.21	4.5	5.5	7.5	11.5
8	256	0	0	0	0	0	0	0	0	0	3.89	5.0	6.0	8.0	12.0
9	512	0	0	0	0	0	0	0	0	0	4.50	5.5	6.5	8.5	12.5
10	1024	0	0	0	0	0	0	0	0	0	6.22	6.0	7.0	9.0	13.0
11	2048	0	0	0	0	0	0	0	0	0	9.57	6.5	7.5	9.5	13.5
12	4096	0	0	0	0	0	0	0	0	0	9.68	7.0	8.0	10.0	14.0
13	8192	0	0	0	0	0	0	0	0	0	6.19	7.5	8.5	10.5	14.5
14	16384	2	0	0	0	0	0	0	0	0	9.55	8.0	9.0	11.0	15.0
15	32768	0	1	0	0	0	0	0	0	0	9.17	8.5	9.5	11.5	15.5
16	65536	**1**	1	0	0	0	0	0	0	0	9.21	9.0	10.0	12.0	16.0
17	131072	2	**3**	0	0	0	0	0	0	0	11.48	9.5	10.5	12.5	16.5
18	262144	3	1	**0**	0	0	0	0	0	0	8.05	10.0	11.0	13.0	17.0
19	524288	5	6	0	**1**	0	0	0	0	0	25.01	10.5	11.5	13.5	17.5
20	1048576	23	8	2	2	**1**	0	0	0	0	14.19	11.0	12.0	14.0	18.0

Table 3: Result of simulating St. Petersburg paradox versus theoretical predictions. The bold numbers indicate the position in the series where theoretically only one game is expected to exist. Furthermore, only one game is expected to appear beyond that position.

from $k+1$ to infinity but additional term beyond the kth term requires an additional flip. Therefore, the probability of the game continuing reduces by half for each term beyond the kth term. Thus the probability that all M games terminate by the $k+1$ flip is 0.5, the probability of all M games terminate by the $k+2$ flip is 0.75, the probability of all M games terminate by the $k+3$ flip is 0.875, and by the $k+4$ flip is 0.9375; and so on. The theoretical expected monetary value of the game can now be determined as a function of M, the number of games played,

$$EMV(M) = \frac{1}{M}\sum(Number\ of\ games \times Payout\ per\ game) \tag{19}$$

Substituting the number of games and the payout, we obtain

$$\begin{aligned} EMV(M) &= \frac{1}{2^k}[2^{k-1}\cdot 2^0 + 2^{k-2}\cdot 2^1 + 2^{k-3}\cdot 2^2 + \ldots + 2^{k-k}\cdot 2^{k-1}] + \frac{f\,p}{2^k} \\ &= \frac{k}{2} + \frac{f\,p}{2^k} \end{aligned} \tag{20}$$

where $f\,p$ is the *final payout.* The contribution of final payout to the EMV is equal to $2^{-k}\cdot 2^k = 1$ if the game terminates at $k+1$ flip, is equal $2^{-k}\cdot 2^k = 2$ if the game terminates at the $k+2$ flip, is equal 4 if the game terminates at $k+3$ flip, and so on. However, the probability decreases at the rate of half for each additional flip of the coin. Then theoretically, the $EMV(M = 2^k) = k/2 + c$ where $c = 1$ at a 50 % confidence level, or $c = 2$ at a 75 % confidence level, or $c = 4$ at an 87.5 % confidence level, and so on. If the game is played only once, $k = 0$ and the $EMV = c$.

It should be noted that the longest run of any game is $k+$ the number flips beyond the kth term; at a 50 % confidence level, the longest run of flips is $k+1$.

The preceding can be illustrated by way of an example. What can one expect if a set of 2^{100} games are played, an enormous number of games? First, the series will consists of 101 terms, not an infinity number of terms, Second, Paul can be 50 % confident that the EMV will be 100/2+1=51 ducats. He can be 75 % confident that it would be 52 ducats or less. He can be 87.5 % confident that it would be 54 ducats or less and so on. Third, at a 50 % confidence level, the longest game will be only 101 flips in duration. Even if the game is played manually, it would take only few minutes to complete. The St. Petersburg game does nor require an infinite amount of time to play. What takes time is the time needed to play the total number of games.

Vivian validates the preceding theory by simulating the St. Petersburg game a number of times and recording the results. The number of games simulated ranges from $k = 0$ (M=1) to $k = 20$ (M=1,048,576). The number of games terminating at each term is recorded. The simulated EMV are then calculated based on the number of games terminating at each term in the series and the payout associated with that term. The results are shown in Table 2 and in Table 3. As expected, the the theoretical and simulated results are in agreement. The empirical results generally hover about the 50 % confidence line; occasionally, the results drift up and touch the 75 % level line and less frequently the results touch the 87.5 % level. The EMV of the simulated results increases as k increases, as one would expect, producing an increasing trend. Overall, the theoretical EMV of the St. Petersburg game is infinite as held by the traditional view, but this can only be achieved if the game is in fact played an infinite number of times, a practical impossibility.

References

[1] I. Berker, E. Csáki, S. Csörgö, *Almost sure limit theorems for the St. Petersburg games*, Statistisc and Probability Letters, 45, 23–30. (1999)

[2] D. Bernoulli, *Specimen theoriae novae de meusura sortis*, Commentarii Academiae Scientiarum Imperialis Petropolitanea, 5, 175–192. (1738) (Traslated and republished in *Exposition of a new theory on the measurement of risk*, Econometrica, 22, 23–36. (1954)

[3] P.R. Blavatskyy, *Back to the St. Petersburg paradox*, Management Science, 51 (4), 667–675. (2005)

[4] G.L.L. Buffon, *Essai d'Arimethmétique Morale*, Supplements à l'Histoire Naturelle, T. IV, Paris. (1777) (Reprinted in *Oeuvres Philosophiques de Buffon*, Paris (1954))

[5] A. Ceasar, *A Monte Carlo Simulation Related to the St. Petersburg Paradox*, College Mathematical Journal, 15 (4) 399-342. (1984)

[6] S. Csörgö, *Merge rates for sums of large winning in generalized St. Petersburg games*, Acta Scientiarum Mathematicarum, 69, 441-454. (2003)

[7] S. Csörgö, G. Simons, *On Steinhaus resolution of the St. Petersburg paradox*, Probability and Mathematical Statistics, 14, 157–172. (1993)

[8] D. Durand, *Growth stocks and the Petersburg paradox*, The Journal of Finance, 12, 348–363. (1957)

[9] W. Feller, *Note on the law of large numbers and fair games*, Annals of Mathematical Statistics, 16, 301–304. (1954)

[10] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, Wiley, New York. (1954)

[11] M.O. Rieger, M. Wang, *Cumulative prospect theory and the St. Petersburg paradox*, Economic Theory, 28, 665–679. (2006)

[12] M.G. Russon, S.J. Chang, *Risk aversion and practical expected value: a simulation of the St. Petersburg game*, Simulation and Gaming, 23 (1), 6–19. (1992)

[13] P.A. Samuelson, *St. Petersburg paradoxes: Defanged, dissected, and historical described*, Journal of Economic Literature, 15, 24–55. (1977)

[14] H. Steinhaus, *The so-called Petersburg paradox*, Colloquium Mathematicum, 2, 56–58. (1949)

[15] G. Székely, D.St.P. Richards, *The St. Petersburg Paradox and the Crash of High-Tech Stocks in 2000*, American Statistical Association, 58 (3), 225–231. (2004)

[16] I. Vardi, *The St. Petersburg paradox and continued fractions*, Comptes Rendus des Séances de l'Academie des Sciences, Sér. I Mathematique, 324, 913–918. (1997)

[17] R.W. Vivian, *Simulating Daniel Bernoulli's St. Petersburg game: Theoretical and empirical consistency*, Simulation and Game, 35, 499-504. (2004)

[18] W.A. Withworth, *Choice and Chance*, Deighton, Bell, & Co., Cambridge. (1901)